# Supplementary Appendix:
# Difference-in-Differences with a Continuous Treatment

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This supplementary appendix provides a number of additional results for "Difference-in-Differences with a Continuous Treatment". Appendix SA presents proof of the results in Theorem 3.4 in Section 3.3 on interpreting TWFE regressions with a continuous treatment. Appendix SB contains more details about the setting with multiple time periods and variation in treatment timing and dose intensity, and it expands upon the results provided in Appendix C in the main text. This section also presents several results on interpreting TWFE regressions in the multiple-period setting. Appendix SC provides proofs for all the results in the main text and in the supplementary appendix concerning multiple periods and variation in treatment timing and dose intensity. Appendix SD provides results and proofs for a number of additional results that were discussed in the main text: results for settings with no untreated units; additional results for TWFE decompositions with a continuous treatment; and TWFE decompositions with a multi-valued discrete treatment. Finally, Appendix SE provides results on relaxing the strong parallel trends assumption, which was briefly discussed in Section 5.1 in the main text.

## SA Proofs of Results from Section 3.3 of the Main Text

This section contains the proofs of the results in Theorem 3.4 in Section 3.3 on interpreting TWFE regressions with a continuous treatment. To conserve on notation, we define

$$m_\Delta(d) = \mathbb{E}[\Delta Y | D = d],$$

We divide the proofs according to each part of the theorem. In the proof, we derive all the results in terms of $m_\Delta(d)$. The result in Theorem 3.4 is stated in terms of various causal building block parameters. Those results follow immediately from the ones below by noting that, under Assumption PT,

- $m_\Delta(d) - m_\Delta(0) = ATT(d|d)$
- $m'_\Delta(d) = ACRT(d|d) + \underbrace{\frac{\partial ATT(d|h)}{\partial h}\Big|_{h=d}}_{\text{selection bias}}$

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- $m_\Delta(h) - m_\Delta(l) = ATT(h|h) - ATT(l|l) = \mathbb{E}[Y_t(h) - Y_t(l)|D = h] + \underbrace{\Big(ATT(l|h) - ATT(l|l)\Big)}_{\text{selection bias}}$

or, when Assumption SPT holds,

- $m_\Delta(d) - m_\Delta(0) = ATT(d)$
- $m'_\Delta(d) = ACRT(d)$
- $m_\Delta(h) - m_\Delta(l) = ATT(h) - ATT(l) = \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D > 0]$

## Proof of Theorem 3.4(a)

*Proof.* First, notice that Equation (1.1) is equivalent to

$$\Delta Y_i = (\theta_{t=2} - \theta_{t=1}) + \beta^{twfe} D_i + \Delta v_{i,t} \tag{S1}$$

which holds by taking first differences and because all units are untreated in the first period. Therefore, it immediately follows that

$$\begin{aligned}
\beta^{twfe} &= \frac{\mathbb{E}[(D - \mathbb{E}[D])\Delta Y]}{\mathrm{Var}(D)} \\
&= \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)}(m_\Delta(D) - m_\Delta(0))\right] \\
&= \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)}(m_\Delta(D) - m_\Delta(0))\Big|D > 0\right]\mathbb{P}(D > 0) \\
&= \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)}(m_\Delta(D) - m_\Delta(d_L))\Big|D > 0\right]\mathbb{P}(D > 0) \\
&\quad + \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)}(m_\Delta(d_L) - m_\Delta(0))\Big|D > 0\right]\mathbb{P}(D > 0) \qquad \text{(S2)} \\
&= A_1 + A_2
\end{aligned}$$

where the first equality holds because Equation (S1) is a simple linear regression of $\Delta Y$ on an intercept and $D$, the second equality holds by the law of iterated expectations and because $\mathbb{E}[(D - \mathbb{E}[D])m_\Delta(0)] = 0$, the third equality holds because $\mathbb{E}[m_\Delta(D) - m_\Delta(0)|D = 0] = 0$, and the fourth equality holds by adding and subtracting $m_\Delta(d_L)$ inside the expectation.

We consider $A_1$ and $A_2$ separately next. First, for $A_1$,

$$\begin{aligned}
A_1 &= \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)}(m_\Delta(D) - m_\Delta(d_L))\Big|D > 0\right]\mathbb{P}(D > 0) \\
&= \frac{\mathbb{P}(D > 0)}{\mathrm{Var}(D)}\int_{d_L}^{d_U}(k - \mathbb{E}[D])(m_\Delta(k) - m_\Delta(d_L))\,dF_{D|D>0}(k) \\
&= \frac{\mathbb{P}(D > 0)}{\mathrm{Var}(D)}\int_{d_L}^{d_U}(k - \mathbb{E}[D])\int_{d_L}^{k} m'_\Delta(l)\,dl\,dF_{D|D>0}(k) \\
&= \frac{\mathbb{P}(D > 0)}{\mathrm{Var}(D)}\int_{d_L}^{d_U}(k - \mathbb{E}[D])\int_{d_L}^{d_U} \mathbf{1}\{l \le k\} m'_\Delta(l)\,dl\,dF_{D|D>0}(k) \\
&= \frac{\mathbb{P}(D > 0)}{\mathrm{Var}(D)}\int_{d_L}^{d_U} m'_\Delta(l)\int_{d_L}^{d_U}(k - \mathbb{E}[D])\mathbf{1}\{l \le k\}\,dF_{D|D>0}(k)\,dl
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{P}(D>0)}{\text{Var}(D)} \int_{d_L}^{d_U} m'_\Delta(l)\mathbb{E}[(D-\mathbb{E}[D])\mathbf{1}\{l \leq D\}|D>0]\,dl \\
&= \frac{\mathbb{P}(D>0)}{\text{Var}(D)} \int_{d_L}^{d_U} m'_\Delta(l)\mathbb{E}[(D-\mathbb{E}[D])|D \geq l]\mathbb{P}(D \geq l|D>0)\,dl \\
&= \int_{d_L}^{d_U} m'_\Delta(l)\frac{(\mathbb{E}[D|D\geq l]-\mathbb{E}[D])\mathbb{P}(D\geq l)}{\text{Var}(D)}\,dl \qquad \text{(S3)}
\end{aligned}
$$

where the first equality is the definition of $A_1$, the second equality holds by rearranging terms and writing the expectation as an integral, the third equality holds by the fundamental theorem of calculus, the fourth equality rewrites the inner integral so that it is over $d_L$ to $d_U$, the fifth equality holds by changing the order of integration and rearranging terms, the sixth equality holds by rewriting the inner integral as an expectation, the seventh equality holds by the law of iterated expectations (and since $D \geq l \implies D > 0$), and the last equality holds by combining terms.

Next, for $A_2$, it immediately holds that

$$
\begin{aligned}
A_2 &= \mathbb{E}\left[\frac{(D-\mathbb{E}[D])}{\text{Var}(D)}(m_\Delta(d_L)-m_\Delta(0))\Big|D>0\right]\mathbb{P}(D>0) \\
&= \frac{(\mathbb{E}[D|D>0]-\mathbb{E}[D])\mathbb{P}(D>0)d_L}{\text{Var}(D)}\frac{(m_\Delta(d_L)-m_\Delta(0))}{d_L} \qquad \text{(S4)}
\end{aligned}
$$

where the first equality is the definition of $A_2$, and the second equality holds by multiplying and dividing by $d_L$.

Then, the first result in Part (a) holds by combining Equations (S3) and (S4). That the weights are all positive holds immediately since $(\mathbb{E}[D|D \geq l] - \mathbb{E}[D]) > 0$ for all $l \geq d_L$, $\mathbb{P}(D \geq l) > 0$ for all $l \geq d_L$, $(\mathbb{E}[D|D>0] - \mathbb{E}[D]) > 0$, $\mathbb{P}(D>0) > 0$, and $\text{Var}(D) > 0$.

Next, we next show that $\int_{d_L}^{d_U} w_1^{acrt}(l)\,dl + w_0^{acrt} = 1$. First, notice that

$$
\begin{aligned}
\int_{d_L}^{d_U} w_1^{acrt}(l)\,dl + w_0^{acrt} &= \frac{1}{\text{Var}(D)}\Bigg\{\int_{d_L}^{d_U} \mathbb{E}[D|D\geq l]\mathbb{P}(D\geq l)\,dl \\
&\qquad - \mathbb{E}[D]\int_{d_L}^{d_U}\mathbb{P}(D\geq l)\,dl \\
&\qquad + \mathbb{E}[D|D>0]\mathbb{P}(D>0)d_L \\
&\qquad - \mathbb{E}[D]\mathbb{P}(D>0)d_L\Bigg\} \\
&= \frac{1}{\text{Var}(D)}\Big\{B_1 - B_2 + B_3 - B_4\Big\}
\end{aligned}
$$

and we consider $B_1, B_2, B_3$, and $B_4$ in turn.

For $B_1$, first notice that for all $l \in \mathcal{D}_+^c$,

$$
\begin{aligned}
\mathbb{E}[D|D\geq l]\mathbb{P}(D\geq l) &= \mathbb{E}[D\mathbf{1}\{D\geq l\}|D\geq l]\mathbb{P}(D\geq l) \\
&= \mathbb{E}[D\mathbf{1}\{D\geq l\}] \qquad \text{(S5)}
\end{aligned}
$$

which holds by the law of iterated expectations and implies that

$$
\begin{aligned}
B_1 &= \int_{d_L}^{d_U} \mathbb{E}[D|D \geq l]\mathbb{P}(D \geq l)\, dl \\
&= \int_{d_L}^{d_U} \int_{\mathcal{D}} d\mathbf{1}\{d \geq l\}\, dF_D(d)\, dl \\
&= \int_{\mathcal{D}} d \left( \int_{d_L}^{d_U} \mathbf{1}\{l \leq d\}\, dl \right) dF_D(d) \\
&= \int_{\mathcal{D}} d(d - d_L)\, dF_D(d) \\
&= \mathbb{E}[D^2] - \mathbb{E}[D]d_L
\end{aligned} \tag{S6}
$$

where the first line is the definition of $B_1$, the second equality holds by Equation (S5), the third equality holds by changing the order of integration, the fourth equality holds by carrying out the inner integration, and the last equality holds by rewriting the integral as an expectation.

Next, for term $B_2$,

$$
\begin{aligned}
B_2 &= \mathbb{E}[D] \int_{d_L}^{d_U} \mathbb{P}(D \geq l)\, dl \\
&= \mathbb{E}[D]\mathbb{P}(D > 0) \int_{d_L}^{d_U} \mathbb{P}(D \geq l|D > 0)\, dl \\
&= \mathbb{E}[D]\mathbb{P}(D > 0) \int_{d_L}^{d_U} \int_{d_L}^{d_U} \mathbf{1}\{d \geq l\}\, dF_{D|D>0}(d)\, dl \\
&= \mathbb{E}[D]\mathbb{P}(D > 0) \int_{d_L}^{d_U} \left( \int_{d_L}^{d_U} \mathbf{1}\{l \leq d\}\, dl \right) dF_{D|D>0}(d) \\
&= \mathbb{E}[D]\mathbb{P}(D > 0) \int_{d_L}^{d_U} (d - d_L)\, dF_{D|D>0}(d) \\
&= \mathbb{E}[D]\mathbb{P}(D > 0)\Big(\mathbb{E}[D|D > 0] - d_L\Big) \\
&= \mathbb{E}[D]^2 - \mathbb{E}[D]\mathbb{P}(D > 0)d_L
\end{aligned} \tag{S7}
$$

where the first equality is the definition of $B_2$, the second equality holds by the law of iterated expectations, the third equality holds by writing $\mathbb{P}(D \geq l|D > 0)$ as an integral, the fourth equality changes the order of integration, the fifth equality carries out the inside integration, the sixth equality rewrites the integral as an expectation, and the last equality holds by combining terms and by the law of iterated expectations.

Next,

$$
\begin{aligned}
B_3 &= \mathbb{E}[D|D > 0]\mathbb{P}(D > 0)d_L \\
&= \mathbb{E}[D]d_L
\end{aligned} \tag{S8}
$$

which holds by the law of iterated expectations. And finally, recall that

$$
B_4 = \mathbb{E}[D]\mathbb{P}(D > 0)d_L \tag{S9}
$$

Thus, from Equations (S6) to (S9), it follows that

$$B_1 - B_2 + B_3 - B_4 = \mathbb{E}[D^2] - \mathbb{E}[D]^2 = \text{Var}(D)$$

which implies the result.

□

## Proof of Theorem 3.4(b)

*Proof.* From the proof of Part (a), we have that

$$\begin{aligned}
\beta^{twfe} &= \frac{\mathbb{P}(D>0)}{\text{Var}(D)}\mathbb{E}\left[(D-\mathbb{E}[D])(m_\Delta(D)-m_\Delta(0))\Big|D>0\right] \\
&= \frac{\mathbb{P}(D>0)}{\text{Var}(D)}\int_{d_L}^{d_U}(l-\mathbb{E}[D])(m_\Delta(l)-m_\Delta(0))\,dF_{D|D>0}(l) \\
&= \frac{1}{\text{Var}(D)}\int_{d_L}^{d_U}(l-\mathbb{E}[D])(m_\Delta(l)-m_\Delta(0))f_D(l)\,dl \\
&= \int_{d_L}^{d_U} w_1^{lev}(l)(m_\Delta(l)-m_\Delta(0))\,dl
\end{aligned}$$

where the second equality holds by writing the expectation as an integral, the third equality holds under Assumption 4(a), and the last equality holds by the definition of $w_1^{lev}$.

Next, we show the properties of the weights for this part of the theorem. The weights can be negative since $l$ can be less than $\mathbb{E}[D]$. To see that the weights integrate to zero, first note that $w_0^{lev}(m_\Delta(0)-m_\Delta(0)) = 0$, so that the previous expression for $\beta^{twfe}$ can equivalently be written as

$$\beta^{twfe} = \int_{d_L}^{d_U} w_1^{lev}(l)(m_\Delta(l)-m_\Delta(0))\,dl + w_0^{lev}(m_\Delta(0)-m_\Delta(0))$$

Then, notice that

$$\begin{aligned}
\int_{d_L}^{d_U} w_1^{lev}(l)\,dl + w_0^{lev} &= \left(\int_{d_L}^{d_U}(l-\mathbb{E}[D])\,dF_D(l) + (0-\mathbb{E}[D])\mathbb{P}(D=0)\right)\Big/\text{Var}(D) \\
&= \left(\int_{\mathcal{D}}(l-\mathbb{E}[D])\,dF_D(l)\right)\Big/\text{Var}(D) \\
&= (\mathbb{E}[D]-\mathbb{E}[D])/\text{Var}(D) \\
&= 0
\end{aligned}$$

where the first equality holds by the definitions of $w_1^{lev}$ and $w_0^{lev}$, the second equality combines terms, and the third and fourth equalities hold immediately. This completes the proof. □

## Proof of Theorem 3.4(c)

*Proof.* From the proof of Theorem 3.4(a), we have that

$$\begin{aligned}
\beta^{twfe} &= \frac{\mathbb{P}(D>0)}{\text{Var}(D)}\mathbb{E}\left[(D-\mathbb{E}[D])(m_\Delta(D)-m_\Delta(0))\Big|D>0\right] \\
&= \frac{\mathbb{P}(D>0)}{\text{Var}(D)}\int_{d_L}^{d_U}(l-\mathbb{E}[D])(m_\Delta(l)-m_\Delta(0))\,dF_{D|D>0}(l)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{P}(D>0)}{\mathrm{Var}(D)} \int_{d_L}^{d_U} (l - \mathbb{E}[D]) l \frac{(m_\Delta(l) - m_\Delta(0))}{l} \, dF_{D|D>0}(l) \\
&= \frac{1}{\mathrm{Var}(D)} \int_{d_L}^{d_U} (l - \mathbb{E}[D]) l \frac{(m_\Delta(l) - m_\Delta(0))}{l} f_D(l) \, dl \\
&= \int_{d_L}^{d_U} w^s(l) \frac{(m_\Delta(l) - m_\Delta(0))}{l} \, dl
\end{aligned}
$$

where the second equality holds by writing the expectation as an integral, the third equality holds by multiplying and dividing by $l$, the fourth equality holds under Assumption 4(a), and the last equality holds by the definition of $w^s$.

The weights can be negative because it is possible that $l < \mathbb{E}[D]$ for some values of $l \in \mathcal{D}_+^c$. That the weights integrate to 1 holds because

$$
\begin{aligned}
\int_{d_L}^{d_U} w^s(l) \, dl &= \left( \int_{d_L}^{d_U} (l - \mathbb{E}[D]) l \, dF_D(l) + (0 - \mathbb{E}[D]) \, 0 \, \mathbb{P}(D=0) \right) \Big/ \mathrm{Var}(D) \\
&= \left( \int_{\mathcal{D}} (l - \mathbb{E}[D]) l \, dF_D(l) \right) \Big/ \mathrm{Var}(D) \\
&= (\mathbb{E}[D^2] - \mathbb{E}[D]^2) \Big/ \mathrm{Var}(D) = 1
\end{aligned}
$$

where the first equality uses the definition of the weights and that $(0 - \mathbb{E}[D]) \, 0 \, \mathbb{P}(D = 0) = 0$, the second equality comes from combining terms, and the last line holds immediately. □

## Proof of Theorem 3.4(d)

*Proof.* From the proof of part (a), we have that

$$
\begin{aligned}
\beta &= \mathbb{E}\left[ \frac{(D - \mathbb{E}[D])}{\mathrm{Var}(D)} m_\Delta(D) \right] \\
&= \frac{1}{\mathrm{Var}(D)} \int_{\mathcal{D}} (h - \mathbb{E}[D]) m_\Delta(h) \, dF_D(h) \\
&= \frac{1}{\mathrm{Var}(D)} \int_{\mathcal{D}} \left( h - \int_{\mathcal{D}} l \, dF_D(l) \right) m_\Delta(h) \, dF_D(h) \\
&= \frac{1}{\mathrm{Var}(D)} \int_{\mathcal{D}} \int_{\mathcal{D}} (h - l) m_\Delta(h) \, dF_D(h) \, dF_D(l) \\
&= \frac{1}{\mathrm{Var}(D)} \int_{\mathcal{D}} \int_{\mathcal{D}, h>l} (h - l)(m_\Delta(h) - m_\Delta(l)) \, dF_D(h) \, dF_D(l) \\
&= \frac{1}{\mathrm{Var}(D)} \int_{\mathcal{D}} \int_{\mathcal{D}, h>l} (h - l)^2 \frac{(m_\Delta(h) - m_\Delta(l))}{(h - l)} \, dF_D(h) \, dF_D(l) \qquad \text{(S10)}
\end{aligned}
$$

where the second equality holds by writing the expectation as an integral, the third equality by writing $\mathbb{E}[D]$ as an integral, the fourth equality rearranges terms, the fifth equality holds because the integrations are symmetric, and the last equality holds by multiplying and dividing by $(h - l)$.

The above arguments hold if the treatment is continuous or discrete. Under Assumption 4(a),

$$
\text{Equation (S10)} = \frac{1}{\mathrm{Var}(D)} \int_{d_L}^{d_U} \int_{\mathcal{D}, h>l} (h - l)^2 \frac{(m_\Delta(h) - m_\Delta(l))}{(h - l)} f_D(h) f_D(l) \, dh \, dl
$$

$$+\frac{1}{\text{Var}(D)}\int_{d_L}^{d_U} h^2 \frac{m_\Delta(h)-m_\Delta(0)}{h} f_D(h)\mathbb{P}(D=0)\,dh$$

which holds by splitting up the first integral in Equation (S10) by whether $l \in \mathcal{D}_+^c$ or $l = 0$. Then, the first part of this result holds by the definition of $w_1^{2\times2}$ and $w_0^{2\times2}$.

That the weights are all positive holds immediately by their definitions. That the weights integrate to one holds because

$$\begin{aligned}\int_{d_L}^{d_U}\int_{\mathcal{D},h>l} w_1^{2\times2,cont}(l,h)\,dh\,dl + \int_{d_L}^{d_U} w_0^{2\times2,cont}(h)\,dh &= \frac{1}{\text{Var}(D)}\int_{\mathcal{D}}\int_{\mathcal{D}} \mathbf{1}\{h>l\}(h-l)^2\,dF_D(h)\,dF_D(l)\\ &= \frac{1}{2}\int_{\mathcal{D}}\int_{\mathcal{D}}(h-l)^2\,dF_D(h)\,dF_D(l)\Big/\text{Var}(D)\\ &= 1\end{aligned}$$

where the first equality holds by combining the integrals and the definition of the weights (it amounts to re-writing the integrals as in Equation (S10)), the second equality holds because $\int_{\mathcal{D}}\int_{\mathcal{D}} \mathbf{1}\{h > l\}(h-l)^2\,dF_D(h)\,dF_D(l) = \int_{\mathcal{D}}\int_{\mathcal{D}} \mathbf{1}\{h \leq l\}(h-l)^2\,dF_D(h)\,dF_D(l)$ (and these two terms add up to the expression on the next line), and the third equality holds because the double integral is equal to $2\text{Var}(D)$. This completes the proof. □

# SB Additional Details for Multiple Periods and Variation in Treatment Timing and Dose

The first part of this section provides some additional identification results for settings with multiple periods. The second part reverse engineers a linear TWFE regression in the case with multiple periods and variation in treatment timing.

## SB.1 Additional Identification Results with Multiple Periods

This section contains two identification results for settings with multiple time periods that supplement the results in Appendix C in the main text and are useful for some later parts of the Supplementary Appendix.

**Theorem S1.** *Under Assumptions 1-MP, 2-MP(a), 3-MP, and SPT-MP, and for all $g \in \bar{\mathcal{G}}$, $t = 2,\ldots,T$ such that $t \geq g$, and for all $d \in \mathcal{D}_+$,*

$$ATT(g,t,d) = \mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] - \mathbb{E}[Y_t - Y_{g-1}|W_t = 0]$$

Theorem S1 complements Theorem C.1 from the main text and shows that, if one invokes strong parallel trends, then the same estimand that identifies $ATT(g,t,d|g,d)$ under parallel trends, identifies $ATT(g,t,d)$.

Finally, for this section, we show that the same sort of selection bias terms as we emphasized in the main text can show up when making comparisons across doses (and, hence, show up in causal response parameters) in a setting with multiple periods and variation in treatment timing and dose under parallel trends assumptions. And, also like in the main text, strong parallel trends can be used to eliminate these selection bias terms.

**Theorem S2.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for all $g \in \bar{\mathcal{G}}$, $t = 2, \ldots, T$ such that $t \geq g$, and for all $d \in \mathcal{D}_+^c$,*

*(1) If, in addition, Assumption PT-MP holds, then*

$$\frac{\partial}{\partial d}\mathbb{E}[Y_t - Y_{g-1}|G = g, D = d] = \frac{\partial}{\partial d}ATT(g,t,d|g,d)$$
$$= ACRT(g,t,d|g,d) + \underbrace{\left.\frac{\partial ATT(g,t,d|g,l)}{\partial l}\right|_{l=d}}_{\text{selection bias}}.$$

*(2) If, in addition, Assumption SPT-MP holds, then*

$$\frac{\partial}{\partial d}\mathbb{E}[Y_t - Y_{g-1}|G = g, D = d] = \frac{\partial}{\partial d}ATT(g,t,d) = ACRT(g,t,d).$$

The proof of Theorem S2 is provided in Appendix SC. Theorem S2 provides an analogous result for the case with multiple periods and variation in treatment timing and dose to Theorems 3.2 and 3.3 in the main text.

## SB.2 TWFE estimators with multiple time periods and variation in treatment timing

In applications with multiple periods and variation in treatment timing and dose, empirical researchers typically estimate the TWFE regression

$$Y_{i,t} = \theta_t + \eta_i + \beta^{twfe} W_{i,t} + v_{i,t}. \tag{S11}$$

where $W_{i,t} = D_i \mathbf{1}\{t \geq G_i\}$. Equation (S11) is the same as the TWFE regression in the baseline case with two periods in Equation (1.1) in the main text, only with the notation slightly adjusted to match the setup of this section. In the main text, we related $\beta^{twfe}$ to several different types of causal effect parameters (see Theorem 3.4 in the main text). In this section, we provide related results for the setting with multiple time periods and variation in treatment timing with a particular emphasis on the comparisons underlying $\beta^{twfe}$ and in causal interpretations (especially causal response interpretations) of $\beta^{twfe}$ in the presence of treatment effect heterogeneity. The results in this section generalize the results in several recent papers on TWFE estimates, including Goodman-Bacon (2021) and de Chaisemartin and D'Haultfoeuille (2020), to our staggered DiD setup with variation in treatment intensity. In this section, we modify our previous notation slightly by setting $G_i = T + 1$ for units that do not participate in the treatment in any period (rather than $G_i = \infty$), which simplifies the exposition in several places in this section.

To start with, write population versions of TWFE adjusted variables as

$$\ddot{W}_{i,t} = (W_{i,t} - \bar{W}_i) - \left(\mathbb{E}[W_t] - \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[W_t]\right), \quad \text{where} \quad \bar{W}_i = \frac{1}{T}\sum_{t=1}^{T} W_{i,t}.$$

The estimand for $\beta^{twfe}$ in Equation (S11) is given by

$$\beta^{twfe} = \frac{\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}Y_{i,t}]}{\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}^2]}. \tag{S12}$$

We present both a "mechanical" decomposition of the TWFE estimator and a "causal" decomposition of the estimand that relates assumptions to interpretation. To define these decompositions, we introduce a bit of new notation. First, define the fraction of periods that units in group $g$ spend treated as

$$\bar{G}_g = \frac{T-(g-1)}{T}.$$

For the untreated group $g = T+1$ so that $\bar{G}_{T+1} = 0$.

Next, we define time periods over which averages are taken. For averaging variables across time periods, we use the following notation, for $t_1 \leq t_2$,

$$\bar{Y}_i^{(t_1,t_2)} = \frac{1}{t_2 - t_1 + 1}\sum_{t=t_1}^{t_2} Y_{i,t}.$$

It is also convenient to define some particular averages across time periods. For two time periods $g$ and $k$, with $k > g$, (below, $g$ and $k$ will often index groups defined by treatment timing), we define

$$\bar{Y}_i^{PRE(g)} = \bar{Y}_i^{(1,g-1)}, \quad \bar{Y}_i^{MID(g,k)} = \bar{Y}_i^{(g,k-1)}, \quad \bar{Y}_i^{POST(k)} = \bar{Y}_i^{(k,T)}, \quad \bar{Y}_i^{POST(g)} = \bar{Y}_i^{(g,T)}.$$

$\bar{Y}_i^{PRE(g)}$ is the average outcome for unit $i$ in periods 1 to $g-1$, $\bar{Y}_i^{MID(g,k)}$ is the average outcome for unit $i$ in periods $g$ to $k-1$, and $\bar{Y}_i^{POST(k)}$ is the average outcome for unit $i$ in periods $k$ to $T$. Below, when $g$ and $k$ index groups, $\bar{Y}_i^{PRE(g)}$ is the average outcome for unit $i$ in periods before units in either group are treated, $\bar{Y}_i^{MID(g,k)}$ is the average outcome for unit $i$ in periods after group $g$ has become treated but before group $k$ has been treated, and $\bar{Y}_i^{POST(k)}$ is the average outcome for unit $i$ after both groups have become treated.

Following Goodman-Bacon (2021), we motivate the decomposition of the TWFE estimand by considering the four types of simple DiD estimands that can be formed using only one source of variation. The first comparison is a within-treatment-group comparison of paths of outcomes among units that experienced different amounts of the treatment.

$$\delta^{WITHIN}(g) = \frac{\mathrm{Cov}(\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}, D|G=g)}{\mathrm{Var}(D|G=g)}. \tag{S13}$$

The second comparison is based on treatment timing. It compares paths of outcomes between a particular timing group $g$ and a "later-treated" group $k$ (i.e., $k > g$) in the periods after group $g$ is treated but before group $k$ becomes treated relative to their common pre-treatment periods.[1]

$$\delta^{MID,PRE}(g,k) = \frac{\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right)|G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right)|G=k\right]}{\mathbb{E}[D|G=g]}. \quad \text{(S14)}$$

Note that this term encompasses comparisons of group $g$ to the "never-treated" group.

The third comparison is between paths of outcomes for the "later-treated" group $k$ in its post-treatment period relative to a pre-treatment period adjusted by the same path of outcomes for the "early-treated" group $g$.

$$\delta^{POST,MID}(g,k) = \frac{\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right)|G=k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right)|G=g\right]}{\mathbb{E}[D|G=k]}. \quad \text{(S15)}$$

It is undesirable that this term shows up in the expression for $\beta^{twfe}$, as it uses the already-treated group $g$ as the comparison group for group $k$.

The final comparison that shows up in the TWFE estimand is between paths of outcomes between "early" and "late" treated groups in their common post-treatment periods relative to their common pre-treatment periods. In other words, this comparison comes from the "endpoints" where the two timing groups are either both untreated or both treated with possibly different average doses.

$$\delta^{POST,PRE}(g,k) = \frac{\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right)|G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right)|G=k\right]}{\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k]}. \quad \text{(S16)}$$

Next, we show how $\beta^{twfe}$ weights these simple DiD terms together and discuss its theoretical interpretation under parallel trends assumptions. To characterize the weights, first, define $p_g = \mathbb{P}(G=g)$ and

$$p_{g|\{g,k\}} = \mathbb{P}(G=g|G \in \{g,k\}).$$

We also define the following weights, which measure the variance of the treatment variable used to estimate each of the simple DiD terms in equations Equations (S13) to (S16).

$$w^{g,within}(g) = \mathrm{Var}(D|G=g)(1-\bar{G}_g)\bar{G}_g p_g \Bigg/ \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}^2],$$

$$w^{g,post}(g,k) = \mathbb{E}[D|G=g]^2(1-\bar{G}_g)(\bar{G}_g - \bar{G}_k)(p_g+p_k)^2 p_{g|\{g,k\}}(1-p_{g|\{g,k\}}) \Bigg/ \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}^2],$$

$$w^{k,post}(g,k) = \mathbb{E}[D|G=k]^2\bar{G}_k(\bar{G}_g - \bar{G}_k)(p_g+p_k)^2 p_{g|\{g,k\}}(1-p_{g|\{g,k\}}) \Bigg/ \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}^2],$$

$$w^{long}(g,k) = (\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])^2\bar{G}_k(1-\bar{G}_g)(p_g+p_k)^2 p_{g|\{g,k\}}(1-p_{g|\{g,k\}}) \Bigg/ \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\ddot{W}_{i,t}^2].$$

Our next result combines the simple DiD terms and their variance weights to provide a mechanical decomposition of $\beta^{twfe}$ in DiD setups with variation in treatment timing and variation in treatment intensity.

[1] Each of the following expressions also includes a term in the denominator. Below, this term is useful for interpreting differences across groups as partial effects of more treatment, but, for now, we largely ignore the expressions in the denominator. That being said, the denominator is important to the definition of each term.

**Proposition S1.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP, $\beta^{twfe}$ in Equation (S11) can be written as*

$$\beta^{twfe} = \sum_{g\in\mathcal{G}} w^{g,within}(g)\delta^{WITHIN}(g)$$
$$+\sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Big\{w^{g,post}(g,k)\delta^{MID,PRE}(g,k) + w^{k,post}(g,k)\delta^{POST,MID}(g,k) + w^{long}(g,k)\delta^{POST,PRE}(g,k)\Big\}.$$

*In addition, (i) $w^{g,within}(g) \geq 0$, $w^{g,post}(g,k) \geq 0$, $w^{k,post}(g,k) \geq 0$, and $w^{long}(g,k) \geq 0$ for all $g \in \mathcal{G}$ and $k \in \mathcal{G}$ with $k > g$, and (ii) $\sum_{g\in\mathcal{G}} w^{g,within}(g) + \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\big\{w^{g,post(g,k)}(g,k) + w^{k,post}(g,k) + w^{long}(g,k)\big\} = 1$.*

Proposition S1 generalizes the decomposition theorem for binary staggered timing designs in Goodman-Bacon (2021) to our setup with variation in treatment intensity. Notice that it does not require Assumption 2-MP(b) and is, therefore, compatible with a binary, multi-valued, continuous, or mixed treatment. It says that $\beta^{twfe}$ can be written as a weighted average of the four comparisons in Equations (S13) to (S16). These weights are all positive and sum to one.

Proposition S1 provides a new, explicit description of what kinds of comparisons TWFE uses to compute $\beta^{twfe}$, but it does not on its own provide guidance on how to interpret TWFE estimates. Next, we analyze the theoretical interpretation of each of these simple DiD estimands under different assumptions and then discuss what this implies about the (arguably implicit) identifying assumptions and estimand for TWFE. We need to introduce additional notation that applies to the underlying causal parameters in the DiD terms in Equations (S13) through (S16):

$$w_1^{within}(g,l) = \frac{\Big(\mathbb{E}[D|G=g, D\geq l] - \mathbb{E}[D|G=g]\Big)}{\text{Var}(D|G=g)}\mathbb{P}(D\geq l|G=g),$$

$$w_1(g,l) = \frac{\mathbb{P}(D\geq l|G=g)}{\mathbb{E}[D|G=g]}, \qquad w_0(g) = \frac{d_L}{\mathbb{E}[D|G=g]},$$

$$w_1^{across}(g,k,l) = \frac{(\mathbb{P}(D\geq l|G=g) - \mathbb{P}(D\geq l|G=k))}{(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])},$$
$$\tilde{w}_1^{across}(g,k,l) = \frac{\mathbb{P}(D\geq l|G=k)}{(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])}, \qquad \tilde{w}_0^{across}(g,k) = \frac{d_L}{(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])}.$$

In addition, define the following differences in paths of outcomes over time

$$\pi^{POST(\tilde{k}),PRE(\tilde{g})}(g) = \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| D=0\right],$$
$$\pi^{MID(\tilde{g},\tilde{k}),PRE(\tilde{g})}(g) = \mathbb{E}\left[\left(\bar{Y}^{MID(\tilde{g},\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(\tilde{g},\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| D=0\right],$$
$$\pi^{POST(\tilde{k}),MID(\tilde{g},\tilde{k})}(g) = \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{MID(\tilde{g},\tilde{k})}\right)\Big| G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{MID(\tilde{g},\tilde{k})}\right)\Big| D=0\right],$$

and, similarly,

$$\pi_D^{POST(\tilde{k}),PRE(\tilde{g})}(g,d) = \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| G=g, D=d\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| D=0\right],$$
$$\pi_D^{MID(\tilde{g},\tilde{k}),PRE(\tilde{g})}(g,d) = \mathbb{E}\left[\left(\bar{Y}^{MID(\tilde{g},\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| G=g, D=d\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(\tilde{g},\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right)\Big| D=0\right],$$
$$\pi_D^{POST(\tilde{k}),MID(\tilde{g},\tilde{k})}(g,d) = \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{MID(\tilde{g},\tilde{k})}\right)\Big| G=g, D=d\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{MID(\tilde{g},\tilde{k})}\right)\Big| D=0\right],$$

which are the same paths of outcomes but conditional on having dose $d$.

The following result is our main result on interpreting TWFE estimates with a continuous treatment.

**Theorem S3.** *Under Assumptions 1-MP, 2-MP, and 3-MP,*

*(1) The four comparisons in Equations (S13) to (S16) can be written as*

$$\delta^{WITHIN}(g) = \int_{d_L}^{d_U} w_1^{within}(g,l) \frac{\partial \pi_D^{POST(g),PRE(g)}(g,l)}{\partial l}\, dl,$$

$$\delta^{MID,PRE}(g,k) = \int_{d_L}^{d_U} w_1(g,l) \frac{\partial \pi_D^{MID(g,k),PRE(g)}(g,l)}{\partial l}\, dl + w_0(g) \frac{\pi_D^{MID(g,k),PRE(g)}(g,d_L)}{d_L} - w_0(g) \frac{\pi^{MID(g,k),PRE(g)}(k)}{d_L},$$

$$\delta^{POST,MID}(g,k) = \int_{d_L}^{d_U} w_1(k,l) \frac{\partial \pi_D^{POST(k),MID(g,k)}(k,l)}{\partial l}\, dl + w_0(k) \frac{\pi^{POST(k),MID(g,k)}(k,d_L)}{d_L} - w_0(k) \left( \frac{\pi^{POST(k),PRE(g)}(g) - \pi^{MID(g,k),PRE(g)}(g)}{d_L} \right),$$

$$\delta^{POST,PRE}(g,k) = \int_{d_L}^{d_U} w_1^{across}(g,k,l) \frac{\partial \pi_D^{POST(k),PRE(g)}(g,l)}{\partial l}\, dl - \left\{ \int_{d_L}^{d_U} \tilde{w}_1^{across}(g,k,l) \left( \frac{\partial \pi_D^{POST(k),PRE(g)}(k,l)}{\partial l} - \frac{\partial \pi_D^{POST(k),PRE(g)}(g,l)}{\partial l} \right) dl + \tilde{w}_0^{across}(g,k) \left( \frac{\pi_D^{POST(k),PRE(g)}(k,d_L) - \pi_D^{POST(k),PRE(g)}(g,d_L)}{d_L} \right) \right\}.$$

*(2) If, in addition, Assumption SPT-MP holds, then*

$$\delta^{WITHIN}(g) = \int_{d_L}^{d_U} w_1^{within}(g,l) \overline{ACRT}^{POST(g)}(g,l) dl,$$

$$\delta^{MID,PRE}(g,k) = \int_{d_L}^{d_U} w_1(g,l) \overline{ACRT}^{MID(g,k)}(g,l)\, dl + w_0(g) \frac{\overline{ATT}^{MID(g,k)}(g,d_L)}{d_L},$$

$$\delta^{POST,MID}(g,k) = \int_{d_L}^{d_U} w_1(k,l) \overline{ACRT}^{POST(k)}(k,l)\, dl + w_0(k) \frac{\overline{ATT}^{POST(k)}(k,d_L)}{d_L} - w_0(k) \left( \frac{\pi^{POST(k),PRE(g)}(g) - \pi^{MID(g,k),PRE(g)}(g)}{d_L} \right),$$

$$\delta^{POST,PRE}(g,k) = \int_{d_L}^{d_U} w_1^{across}(g,k,l) \overline{ACRT}^{POST(k)}(g,l)\, dl - \left\{ \int_{d_L}^{d_U} \tilde{w}_1^{across}(g,k,l) \left( \frac{\partial \pi_D^{POST(k),PRE(g)}(k,l)}{\partial l} - \frac{\partial \pi_D^{POST(k),PRE(g)}(g,l)}{\partial l} \right) dl + \tilde{w}_0^{across}(g,k) \left( \frac{\pi_D^{POST(k),PRE(g)}(k,d_L) - \pi_D^{POST(k),PRE(g)}(g,d_L)}{d_L} \right) \right\}.$$

*In addition, (i)* $w_1^{within}(g,d) \geq 0$*,* $w_1(g,d) \geq 0$*, and* $w_0(g) \geq 0$*, for all* $g \in \mathcal{G}$ *and* $d \in \mathcal{D}_+^c$ *and (ii)* $\int_{d_L}^{d_U} w_1^{within}(g,l)\, dl = 1$*,* $\int_{d_L}^{d_U} w_1(g,l)\, dl + w_0(g) = 1$*, and* $\int_{d_L}^{d_U} w_1^{across}(g,k,l)\, dl = 1$*.*

Part (1) of Theorem S3 links the four sets of comparisons in the TWFE estimator in Proposition S1 to derivatives of conditional expectations along with some additional (nuisance) paths of outcomes. Part (2) of Theorem S3 imposes the multi-period version of strong parallel trends in Assumption SPT-MP. Under Assumption SPT-MP, $\delta^{WITHIN}(g)$ and $\delta^{MID,PRE}(g,k)$ both deliver weighted averages of $ACRT$-type parameters. However, $\delta^{POST,MID}(g,k)$ and $\delta^{POST,PRE}(g,k)$ still involve non-negligible nuisance terms. Under Assumption SPT-MP, the additional term in $\delta^{POST,MID}(g,k)$ involves the difference between treatment effects for group $g$ in group $k$'s post-treatment periods relative to treatment effects for group $g$ in the periods after group $g$ is treated but before group $k$ is treated—that is, treatment effect dynamics. Parallel trends assumptions do not imply that this term is equal to 0. And, in the special case where the treatment is binary, this term corresponds to the "problematic" term related to treatment effect dynamics in Goodman-Bacon (2021).

The additional nuisance term in $\delta^{POST,PRE}(g,k)$ involves differences in partial effects of more treatment across groups in their common post-treatment periods. Parallel trends does not restrict these partial effects to be equal to each other. This term does not show up in the case with a binary treatment because, by construction, the distribution of the dose is the same across groups. It is helpful to further consider where this expression comes from. For simplicity, temporarily suppose that the partial effect of more dose is positive and constant across groups, time, and dose. In this case, if group $g$ has more dose on average than group $k$, then its outcomes should increase more from group $g$ and $k$'s common pre-treatment period to their common post-treatment period. This is the comparison that shows up in $\delta^{POST,PRE}(g,k)$. However, when partial effects are not the same across groups and times (which is not implied by any parallel trends assumption), then, for example, it could be the case that the partial effect of dose is positive for all groups and time periods but greater for group $k$ relative to group $g$. If these differences are large enough, it could lead to the cross-group, long-difference comparisons in $\delta^{POST,PRE}(g,k)$ having the opposite sign.

Next, we discuss what sort of extra conditions can (i) guarantee that $\beta^{twfe}$ is a (positively) weighted average of underlying causal responses or (ii) for $\beta^{twfe} = ACRT^{\texttt{glob}} := \mathbb{E}\Big[ACRT^{dose}(D)\Big| G \leq T\Big]$, i.e., the overall average causal response.[2] To do so, we introduce restrictions on different types of treatment effect heterogeneity.

**Assumption S1** (Assumptions Limiting Treatment Effect Heterogeneity)**.**

*(a) [No Treatment Effect Dynamics] For all* $g \in \bar{\mathcal{G}}$ *and* $t \geq g$ *(i.e, post-treatment periods for group* $g$*),* $ACRT(g,t,d)$ *and* $ATT(g,t,d_L)$ *do not vary with* $t$*.*

*(b) [Homogeneous Causal Responses across Groups] For all* $g \in \bar{\mathcal{G}}$ *with* $t \geq g$ *and* $k \in \bar{\mathcal{G}}$ *with* $t \geq k$*,* $ACRT(g,t,d) = ACRT(k,t,d)$ *and* $ATT(g,t,d_L) = ATT(k,t,d_L)$*.*

*(c) [Homogeneous Causal Responses across Dose] For all* $g \in \bar{\mathcal{G}}$ *with* $t \geq g$*,* $ACRT(g,t,d)$ *does not vary across* $d$*, and, in addition,* $ATT(g,t,d_L)/d_L = ACRT(g,t,d)$*.*

[2] $ACRT^{dose}(d)$ is the "global" version of $ACRT^{dose}(d|d)$ from the main text. It is defined as $\frac{\partial ATT^{dose}(d)}{\partial d}$ where $ATT^{dose}(d) := \mathbb{E}[\overline{TE}(d)|G \leq T]$.

Assumption S1 introduces three additional conditions limiting treatment effect heterogeneity. Assumption S1(a) imposes that, within a timing-group, the causal response to the treatment does not vary across time, which rules out treatment effect dynamics. Assumption S1(b) imposes that, for a fixed time period, causal responses to the treatment are constant across timing-groups. Assumption S1(c) imposes that, within timing-group and time period, the causal response to more dose is constant across different values of the dose.

**Proposition S2.** *Under Assumptions 1-MP, 2-MP, 3-MP, and SPT-MP,*

*(a) If, in addition, Assumption S1(a) holds, then*

$$\delta^{POST,MID}(g,k) = \int_{d_L}^{d_U} w_1(k,l)\overline{ACRT}^{POST(k)}(k,l)\,dl + w_0(k)\frac{\overline{ATT}^{POST(k)}(k,d_L)}{d_L}.$$

*(b) If, in addition, Assumption S1(b) holds, then*

$$\delta^{POST,PRE}(g,k) = \int_{d_L}^{d_U} w_1^{across}(g,k,l)\overline{ACRT}^{POST(k)}(g,l)\,dl.$$

*(c) If, in addition, Assumption S1(a), (b), and (c) hold, then*

$$\beta^{twfe} = ACRT^{glob}.$$

Proposition S2 provides additional conditions under which the nuisance terms in $\delta^{POST,MID}(g,k)$ and $\delta^{POST,PRE}(g,k)$ are equal to 0. For $\delta^{POST,MID}(g,k)$, these nuisance terms will be equal to 0 if there are no treatment effect dynamics; that is, the causal response to more dose does not vary across time. Ruling out these sorts of treatment effect dynamics is analogous to the kinds of conditions that are required to rule out negative weights in TWFE estimates with a binary treatment. For $\delta^{POST,PRE}(g,k)$, the nuisance terms will be equal to 0 if there are homogeneous causal responses across groups—that the causal response to more dose is the same across groups conditional on having the same amount of dose and being in the same time period. Neither of these assumptions is implied by any of the parallel trends assumptions that we have considered, and they are both potentially very strong. Therefore, under both Assumption S1(a) and (b), $\beta^{twfe}$ is equal to a weighted average of average causal response parameters, but these weights continue to be driven by the TWFE estimation strategy and, like in the baseline two-period case, can continue to deliver poor estimates of the overall average causal response to the treatment. If all of the conditions in Assumption S1(a), (b), and (c) hold, then it implies that $ACRT(g,t,d)$ does not vary by timing group, time period, or the amount of dose, and part (c) of Proposition S2 says that $\beta^{twfe}$ is equal to the overall average causal response under these additional, strong conditions.

# SC Proofs of Results from Appendix C and Appendix SB

This section contains the proofs of results from Appendix C and Appendix SB, which encompass our results on DiD with a continuous treatment and with multiple periods and variation in treatment timing and dose intensity.

## SC.1 Proof of Results from Appendix C

This section proves Theorem C.1, Theorem S1, and Theorem S2.

### Proof of Theorem C.1

*Proof.* For the first part, notice that

$$\begin{aligned}
ATT(g,t,d|g,d) &= \mathbb{E}[Y_t(g,d) - Y_t(0)|G=g, D=d] \\
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(0)|G=g, D=d] - \mathbb{E}[Y_t(0) - Y_{g-1}(0)|G=g, D=d] \\
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(0)|G=g, D=d] - \sum_{s=g}^{t} \mathbb{E}[Y_s(0) - Y_{s-1}(0)|G=g, D=d] \\
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(0)|G=g, D=d] - \sum_{s=g}^{t} \mathbb{E}[Y_s(0) - Y_{s-1}(0)|W_t=0] \\
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(0)|G=g, D=d] - \mathbb{E}[Y_t(0) - Y_{g-1}(0)|W_t=0] \\
&= \mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] - \mathbb{E}[Y_t - Y_{g-1}|W_t=0]
\end{aligned}$$

where the first equality is the definition of $ATT(g,t,d|g,d)$, the second equality holds by adding and subtracting $\mathbb{E}[Y_{g-1}(0)|G=g,D=d]$, the third equality holds by adding and subtracting $\mathbb{E}[Y_s(0)|G=g,D=d]$ for $s=g,\dots,(t-1)$, the fourth equality holds under Assumption PT-MP, the fifth equality holds by canceling all the terms involving $\mathbb{E}[Y_s(0)|W_t=0]$ for $s=g,\dots,(t-1)$ (i.e., from the reverse of the argument for the third equality), and the last equality holds from writing the potential outcomes in terms of their observed counterparts.

For the second part, notice that

$$\begin{aligned}
ACRT(g,t,d|g,d) &= \frac{\partial}{\partial l}\Big\{\mathbb{E}\left[Y_t(g,l)|G=g,D=d\right]\Big\}\Big|_{l=d} \\
&= \frac{\partial}{\partial l}\Big\{\mathbb{E}\left[Y_t(g,l) - Y_{g-1}(0)|G=g,D=d\right]\Big\}\Big|_{l=d} \\
&= \frac{\partial}{\partial l}\Big\{\mathbb{E}\left[Y_t(g,l) - Y_{g-1}(0)|G=g,D=l\right]\Big\}\Big|_{l=d} \\
&= \frac{\partial \mathbb{E}[Y_t - Y_{g-1}|G=g,D=d]}{\partial d}
\end{aligned}$$

where the first equality holds by the definition of $ACRT(g,t,d|g,d)$, the second equality holds by subtracting $\mathbb{E}[Y_{g-1}(0)|G=g,D=d]$ (which does not depend on $l$ and, hence, has zero derivative), the third equality holds by Assumption SPT-MP, and the last equality holds by replacing potential outcomes with their observed counterpart and evaluating the partial derivative at $l=d$. □

### Proof of Theorem S1

*Proof.* Notice that,

$$ATT(g,t,d) = \mathbb{E}[Y_t(g,d) - Y_t(0)|G=g]$$

$$
\begin{aligned}
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(g,d)|G=g] - \mathbb{E}[Y_t(0) - Y_{g-1}(0)|G=g] \\
&= \sum_{s=g}^{t} \mathbb{E}[Y_s(g,d) - Y_{s-1}(g,d)|G=g] - \sum_{s=g}^{t} \mathbb{E}[Y_s(0) - Y_{s-1}(0)|G=g] \\
&= \sum_{s=g}^{t} \mathbb{E}[Y_s(g,d) - Y_{s-1}(g,d)|G=g, D=d] - \sum_{s=g}^{t} \mathbb{E}[Y_s(0) - Y_{s-1}(0)|W_t=0] \\
&= \mathbb{E}[Y_t(g,d) - Y_{g-1}(g,d)|G=g, D=d] - \mathbb{E}[Y_t(0) - Y_{g-1}(0)|W_t=0] \\
&= \mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] - \mathbb{E}[Y_t - Y_{g-1}|W_t=0]
\end{aligned}
$$

where the first equality holds by the definition of $ATT(g,t,d)$, the second equality adds and subtracts $\mathbb{E}[Y_{g-1}(g,d)|G=g]$ (this equation also uses the no anticipation condition in Assumption 3-MP which implies that $\mathbb{E}[Y_{g-1}(g,d)|G=g] = \mathbb{E}[Y_{g-1}(0)|G=g]$), the third equality holds by writing both "long differences" as summations over "short differences", the fourth equality holds by Assumption SPT-MP, the fifth equality holds by canceling all of the intermediate terms in the summations over short differences, and the last equality holds by writing potential outcomes in terms of their corresponding observed outcomes and is the result. □

**Proof of Theorem S2**

*Proof.* To start with, notice that

$$
\frac{\partial}{\partial d}\mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] = \frac{\partial}{\partial d}\Big\{\mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] - \mathbb{E}[Y_t - Y_{g-1}|W_t=0]\Big\} \tag{S17}
$$

which holds because the second term does not depend on $d$. Thus, under Assumption PT-MP, we have that

$$
\begin{aligned}
\frac{\partial}{\partial d}\mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] &= \frac{\partial}{\partial d} ATT(g,t,d|g,d) \\
&= ACRT(g,t,d|g,d) + \frac{\partial ATT(g,t,d|g,l)}{\partial l}\bigg|_{l=d}
\end{aligned}
$$

where the first equality holds by Equation (S17) and Theorem C.1, and the second equality holds by the linearity of differentiation and the definition of $ACRT(g,t,d|g,d)$.

Under Assumption SPT-MP, we have that

$$
\begin{aligned}
\frac{\partial}{\partial d}\mathbb{E}[Y_t - Y_{g-1}|G=g, D=d] &= \frac{\partial}{\partial d} ATT(g,t,d) \\
&= ACRT(g,t,d)
\end{aligned}
$$

where the first equality holds by Equation (S17) and Theorem S1, and the second equality holds by the definition of $ACRT(g,t,d)$. This completes the proof. □

## SC.2 Proofs of Results from Appendix SB.2

This section contains the proofs for interpreting TWFE regressions in the case with a continuous treatment, multiple periods, and variation in treatment timing as in Appendix SB.2.

Before proving the main results in this section, we introduce some additional notation. Let

$$v(g,t) = \mathbf{1}\{t \geq g\} - \bar{G}_g \tag{S18}$$

where the term $\mathbf{1}\{t \geq g\}$ is equal to one in post-treatment time periods for units in group $g$ and recalling that we defined $\bar{G}_g = \frac{T-g+1}{T}$ which is the fraction of periods that units in group $g$ are exposed to the treatment (and notice that this latter term does not depend on the particular time period $t$). Further, notice that $v(g,t)$ is positive in post-treatment time periods and negative in pre-treatment time periods for units in a particular group. Finally, also note that, for the "never-treated" group, $g = T+1$, so that both terms in the expression for $v$ are equal to 0.

Furthermore, recall that, for $1 \leq t_1 \leq t_2 \leq T$, we defined

$$\bar{Y}_i^{(t_1,t_2)} = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} Y_{i,t}$$

where below (and following the notation used throughout the paper), we sometimes leave the subscript $i$ implicit.

We next state and prove some additional results that are helpful for proving the main results. The first lemma rewrites (overall) expected dose experienced in period $t$ adjusted by the overall expected dose (across periods and units) in a form that is useful in proving later results.

**Lemma S1.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] = \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d v(g,t) \, dF_{D|G}(d|g) p_g$$

*Proof.* First, notice that

$$\begin{aligned}
\mathbb{E}[W_t] &= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} \mathbb{E}[W_t | G=g, D=d] \, dF_{D|G}(d|g) p_g \\
&= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\mathbf{1}\{t \geq g\} \, dF_{D|G}(d|g) p_g
\end{aligned} \tag{S19}$$

where the first equality holds by the law of iterated expectations and the second equality holds because, after conditioning on group and dose, $W_t$ is fully determined. Thus,

$$\begin{aligned}
\mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] &= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\mathbf{1}\{t \geq g\} \, dF_{D|G}(d|g) p_g - \frac{1}{T}\sum_{s=1}^{T} \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\mathbf{1}\{s \geq g\} \, dF_{D|G}(d|g) p_g \\
&= \frac{1}{T}\sum_{s=1}^{T} \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\left(\mathbf{1}\{t \geq g\} - \mathbf{1}\{s \geq g\}\right) \, dF_{D|G}(d|g) p_g \\
&= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d \left\{ \frac{1}{T}\sum_{s=1}^{T} \mathbf{1}\{t \geq g\} - \mathbf{1}\{s \geq g\} \right\} dF_{D|G}(d|g) p_g \\
&= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d \left\{ \mathbf{1}\{t \geq g\} - \frac{T-g+1}{T} \right\} dF_{D|G}(d|g) p_g
\end{aligned}$$

$$= \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d v(g,t) \, dF_{D|G}(d|g) p_g$$

where the first equality applies Equation (S19) to both terms, the second equality combines terms by averaging the first term across time periods, the third equality re-orders the summations/integrals, the fourth equality holds because $\mathbf{1}\{t \geq g\}$ does not depend on $s$ and by counting the fraction of periods where $s \geq g$, and the last equality holds by the definition of $v(g,t)$. □

The next lemma provides an intermediate result for the expression for the numerator of $\beta^{twfe}$ in Equation (S12).

**Lemma S2.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\ddot{W}_{i,t} Y_{i,t}] = \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d \left( \mathbb{E}[Y_t | G=g, D=d] - \mathbb{E}[Y_t] \right) v(g,t) \, dF_{D|G}(d|g) p_g \right\}$$

*Proof.* Starting with the term on the left-hand side, we have that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\ddot{W}_{i,t} Y_{i,t}] \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \mathbb{E}[W_{i,t} Y_{i,t}] - \mathbb{E}[\bar{W}_i Y_{i,t}] - \left( \mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] \right) \mathbb{E}[Y_t] \right\} \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \mathbb{E}[D\mathbf{1}\{t \geq G\} Y_t] - \mathbb{E}\left[ D \frac{T-G+1}{T} Y_t \right] - \left( \mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] \right) \mathbb{E}[Y_t] \right\} \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} \left( \mathbb{E}[d\mathbf{1}\{t \geq g\} Y_t | G=g, D=d] - \mathbb{E}\left[ d \frac{T-g+1}{T} Y_t \Big| G=g, D=d \right] \right) dF_{D|G}(d|g) p_g \right. \\
&\qquad \left. - \left( \mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] \right) \mathbb{E}[Y_t] \right\} \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\Big( \mathbb{E}[Y_t | G=g, D=d] v(g,t) \Big) dF_{D|G}(d|g) p_g - \left( \mathbb{E}[W_t] - \frac{1}{T}\sum_{s=1}^{T} \mathbb{E}[W_s] \right) \mathbb{E}[Y_t] \right\} \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\Big( \mathbb{E}[Y_t | G=g, D=d] v(g,t) \Big) dF_{D|G}(d|g) p_g - \left( \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d v(g,t) \, dF_{D|G}(d|g) p_g \right) \mathbb{E}[Y_t] \right\} \\
&= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d \left( \mathbb{E}[Y_t | G=g, D=d] - \mathbb{E}[Y_t] \right) v(g,t) \, dF_{D|G}(d|g) p_g \right\}
\end{aligned}$$

where the first equality holds by the definition of $\ddot{W}_{i,t}$, the second equality holds by plugging in for $W_{i,t}$ and $\bar{W}_i$, the third equality holds by the law of iterated expectations, the fourth equality holds by the definition of $v(g,t)$, the fifth equality holds by Lemma S1, and the sixth equality combines terms. □

Next, based on the result in Lemma S2, we can write the numerator of $\beta^{twfe}$ as

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\ddot{W}_{i,t} Y_{i,t}]$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\left(\mathbb{E}[Y_t|G=g,D=d]-\mathbb{E}[Y_t]\right)v(g,t)\,dF_{D|G}(d|g)p_g\right\}$$

$$= \frac{1}{T}\sum_{t=1}^{T}\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g,D=d]-\mathbb{E}[Y_t|G=g]\Big)v(g,t)\,dF_{D|G}(d|g)p_g \tag{S20}$$

$$+ \frac{1}{T}\sum_{t=1}^{T}\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t]\Big)v(g,t)\,dF_{D|G}(d|g)p_g \tag{S21}$$

where the first equality holds from Lemma S2 and the second equality holds by adding and subtracting $\mathbb{E}[Y_t|G=g]$.

The expression in Equation (S20) involves comparisons between units in the same timing group but that have different doses. The expression in Equation (S21) involves comparisons across different timing groups. We consider each of these terms in more detail below.

**Lemma S3.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g,D=d]-\mathbb{E}[Y_t|G=g]\Big)v(g,t)\,dF_{D|G}(d|g)p_g$$
$$= \sum_{g\in\mathcal{G}}\left\{(1-\bar{G}_g)\bar{G}_g\mathrm{Cov}\left(\bar{Y}^{POST(g)}-\bar{Y}^{PRE(g)}, D\,\Big|\, G=g\right)\right\}p_g$$

*Proof.* Notice that

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g,D=d]-\mathbb{E}[Y_t|G=g]\Big)v(g,t)\,dF_{D|G}(d|g)p_g$$
$$= \frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{g\in\mathcal{G}}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]v(g,t)p_g\right\}$$
$$= \sum_{g\in\mathcal{G}}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]v(g,t)\right\}p_g$$
$$= \sum_{g\in\mathcal{G}}\left\{-\frac{1}{T}\frac{(T-g+1)}{T}\sum_{t=1}^{g-1}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]\right.$$
$$\left.+\frac{1}{T}\frac{(g-1)}{T}\sum_{t=g}^{T}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]\right\}p_g$$
$$= \sum_{g\in\mathcal{G}}\left\{\frac{g-1}{T}\frac{(T-g+1)}{T}\left(\frac{1}{T-g+1}\sum_{t=g}^{T}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]\right.\right.$$
$$\left.\left.-\frac{1}{g-1}\sum_{t=1}^{g-1}\mathbb{E}[Y_t(D-\mathbb{E}[D|G=g])|G=g]\right)\right\}p_g$$
$$= \sum_{g\in\mathcal{G}}\left\{\frac{g-1}{T}\frac{(T-g+1)}{T}\left(\mathbb{E}\big[\big(\bar{Y}^{POST(g)}-\bar{Y}^{PRE(g)}\big)(D-\mathbb{E}[D|G=g])\big|G=g\big]\right)\right\}p_g$$

$$= \sum_{g \in \mathcal{G}} \left\{ (1 - \bar{G}_g)\bar{G}_g \left( \mathbb{E}\Big[ \big( \bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)} \big) (D - \mathbb{E}[D|G=g]) \Big| G = g \Big] \right) \right\} p_g$$

$$= \sum_{g \in \mathcal{G}} \left\{ (1 - \bar{G}_g)\bar{G}_g \text{Cov} \left( \bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}, D \,\Big|\, G = g \right) \right\} p_g$$

where the first equality holds by the law of iterated expectations (and combining terms involving $d$ and $Y_t$), the second equality changes the order of the summations, the third equality holds by splitting the summation involving $t$ in time period $g$ and plugs in for $v(g,t)$ (which is constant within group $g$ and across time periods from $1, \ldots, g-1$ and from $g, \ldots, T$), the fourth equality multiplies and divides by terms so that the inside expressions can be written as averages, the fifth equality holds by changing the order of the expectation and averaging over time periods, the sixth equality holds by the definition of $\bar{G}_g$, and the last equality holds by the definition of covariance. □

Next, we consider the expression from Equation (S21) above, which arises from differences in outcomes across groups. We handle this term over several following results.

**Lemma S4.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t] \Big) v(g,t)\, dF_{D|G}(d|g) p_g \right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k > g} \Big( \mathbb{E}[D|G=g]v(g,t) - \mathbb{E}[D|G=k]v(k,t) \Big) \Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t|G=k] \Big) p_k p_g \right\}$$

*Proof.* Notice that

$$\frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \int_{\mathcal{D}} d\Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t] \Big) v(g,t)\, dF_{D|G}(d|g) p_g \right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \mathbb{E}[D|G=g] \Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t] \Big) v(g,t) p_g \right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \mathbb{E}[D|G=g] \Big( \mathbb{E}[Y_t|G=g] - \sum_{k \in \mathcal{G}} \mathbb{E}[Y_t|G=k] p_k \Big) v(g,t) p_g \right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}} \mathbb{E}[D|G=g] v(g,t) \Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t|G=k] \Big) p_k p_g \right\}$$
$$= \frac{1}{T}\sum_{t=1}^{T} \left\{ \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k > g} \Big( \mathbb{E}[D|G=g]v(g,t) - \mathbb{E}[D|G=k]v(k,t) \Big) \Big( \mathbb{E}[Y_t|G=g] - \mathbb{E}[Y_t|G=k] \Big) p_k p_g \right\}$$

where the first equality holds by integrating over $\mathcal{D}$, the second equality holds by the law of iterated expectations, the third equality holds by combining terms, and the last equality holds because all combinations of $g$ and $k$ occur twice. □

Lemma S4 is helpful because it shows that the cross-group part of the TWFE estimator can be written as comparisons for each group relative to later-treated groups.

Next, we provide an important intermediate result. Before stating this result, notice that

$$\mathbb{E}[D|G=g]v(g,t) - \mathbb{E}[D|G=k]v(k,t)$$

$$= \begin{cases} -\mathbb{E}[D|G=g]\bar{G}_g + \mathbb{E}[D|G=k]\bar{G}_k & \text{for } t < g < k \\ \mathbb{E}[D|G=g](1-\bar{G}_g) + \mathbb{E}[D|G=k]\bar{G}_k & \text{for } g \leq t < k \\ \mathbb{E}[D|G=g](1-\bar{G}_g) - \mathbb{E}[D|G=k](1-\bar{G}_k) & \text{for } g < k \leq t \end{cases} \tag{S22}$$

which holds by the definition of $v$ and is useful for the proof of the following lemma.

**Lemma S5.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t]\Big)v(g,t)\,dF_{D|G}(d|g)p_g\right\}$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{\mathbb{E}[D|G=g](1-\bar{G}_g)(\bar{G}_g-\bar{G}_k)\Bigg(\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=k\right]\Bigg)$$
$$+\mathbb{E}[D|G=k]\bar{G}_k(\bar{G}_g-\bar{G}_k)\Bigg(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=k\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=g\right]\Bigg)$$
$$+(\mathbb{E}[D|G=g]-\mathbb{E}[D|G=k])\bar{G}_k(1-\bar{G}_g)\Bigg(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=k\right]\Bigg)\Bigg\}p_kp_g$$

*Proof.* The result holds as follows

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{g\in\mathcal{G}}\int_{\mathcal{D}} d\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t]\Big)v(g,t)\,dF_{D|G}(d|g)p_g\right\}$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\left\{\frac{1}{T}\sum_{t=1}^{T}\Big(\mathbb{E}[D|G=g]v(g,t)-\mathbb{E}[D|G=k]v(k,t)\Big)\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t|G=k]\Big)\right\}p_kp_g$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{\frac{1}{T}\left(-\mathbb{E}[D|G=g]\bar{G}_g+\mathbb{E}[D|G=k]\bar{G}_k\right)\sum_{t=1}^{g-1}\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t|G=k]\Big)$$
$$+\frac{1}{T}\left(\mathbb{E}[D|G=g](1-\bar{G}_g)+\mathbb{E}[D|G=k]\bar{G}_k\right)\sum_{t=g}^{k-1}\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t|G=k]\Big)$$
$$+\frac{1}{T}\left(\mathbb{E}[D|G=g](1-\bar{G}_g)-\mathbb{E}[D|G=k](1-\bar{G}_k)\right)\sum_{t=k}^{T}\Big(\mathbb{E}[Y_t|G=g]-\mathbb{E}[Y_t|G=k]\Big)\Bigg\}p_kp_g$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{(1-\bar{G}_g)\left(-\mathbb{E}[D|G=g]\bar{G}_g+\mathbb{E}[D|G=k]\bar{G}_k\right)\Big(\mathbb{E}[\bar{Y}^{PRE(g)}|G=g]-\mathbb{E}[\bar{Y}^{PRE(g)}|G=k]\Big)$$
$$+(\bar{G}_g-\bar{G}_k)\left(\mathbb{E}[D|G=g](1-\bar{G}_g)+\mathbb{E}[D|G=k]\bar{G}_k\right)\Big(\mathbb{E}[\bar{Y}^{MID(g,k)}|G=g]-\mathbb{E}[\bar{Y}^{MID(g,k)}|G=k]\Big)$$
$$+\bar{G}_k\left(\mathbb{E}[D|G=g](1-\bar{G}_g)-\mathbb{E}[D|G=k](1-\bar{G}_k)\right)\Big(\mathbb{E}[\bar{Y}^{POST(k)}|G=g]-\mathbb{E}[\bar{Y}^{POST(k)}|G=k]\Big)\Bigg\}p_kp_g$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{(1-\bar{G}_g)\left(-\mathbb{E}[D|G=g](\bar{G}_g-\bar{G}_k)+(\mathbb{E}[D|G=k]-\mathbb{E}[D|G=g])\bar{G}_k\right)\Big(\mathbb{E}[\bar{Y}^{PRE(g)}|G=g]-\mathbb{E}[\bar{Y}^{PRE(g)}|G=k]\Big)$$
$$+(\bar{G}_g-\bar{G}_k)\left(\mathbb{E}[D|G=g](1-\bar{G}_g)+\mathbb{E}[D|G=k]\bar{G}_k\right)\Big(\mathbb{E}[\bar{Y}^{MID(g,k)}|G=g]-\mathbb{E}[\bar{Y}^{MID(g,k)}|G=k]\Big)$$
$$+\bar{G}_k\left((\mathbb{E}[D|G=g]-\mathbb{E}[D|G=k])(1-\bar{G}_g)-\mathbb{E}[D|G=k](\bar{G}_g-\bar{G}_k)\right)\Big(\mathbb{E}[\bar{Y}^{POST(k)}|G=g]-\mathbb{E}[\bar{Y}^{POST(k)}|G=k]\Big)\Bigg\}p_kp_g$$

$$= \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{\mathbb{E}[D|G=g](1-\bar{G}_g)(\bar{G}_g-\bar{G}_k)\Bigg(\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=k\right]\Bigg)$$
$$+\mathbb{E}[D|G=k]\bar{G}_k(\bar{G}_g-\bar{G}_k)\Bigg(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=k\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=g\right]\Bigg)$$
$$+(\mathbb{E}[D|G=g]-\mathbb{E}[D|G=k])\bar{G}_k(1-\bar{G}_g)\Bigg(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=k\right]\Bigg)\Bigg\}p_kp_g$$

where the first equality uses the result in Lemma S4, the second equality changes the order of the summations (splitting them at $g$ and $k$ where the values of $v(g,t)$ and $v(k,t)$ change) and uses

Equation (S22), the third equality holds by averaging over time periods (which involves multiplying and dividing by $g-1$ in the first line, multiplying and dividing by $k-g$ in the second line, and multiplying and dividing by $T-k+1$ in the last line), the fourth equality rearranges the expressions for the weights, and the fifth equality holds by rearranging terms with common weights. □

Define the following weights

$$\begin{aligned}
\tilde{w}^{g,within}(g) &= \text{Var}(D|G=g)(1-\bar{G}_g)\bar{G}_g p_g \\
\tilde{w}^{g,post}(g,k) &= \mathbb{E}[D|G=g]^2(1-\bar{G}_g)(\bar{G}_g-\bar{G}_k)p_k p_g \\
\tilde{w}^{k,post}(g,k) &= \mathbb{E}[D|G=k]^2\bar{G}_k(\bar{G}_g-\bar{G}_k)p_k p_g \\
\tilde{w}^{long}(g,k) &= (\mathbb{E}[D|G=g]-\mathbb{E}[D|G=k])^2\bar{G}_k(1-\bar{G}_g)p_k p_g
\end{aligned}$$

which correspond to $w^{g,within}(g)$, $w^{g,post}(g,k)$, $w^{k,post}(g,k)$, and $w^{long}(g,k)$ above except they do not divide by $T^{-1}\sum_{t=1}^T \mathbb{E}[\ddot{W}_{i,t}^2]$.[3] The next result provides a decomposition of the numerator of $\beta^{twfe}$.

**Lemma S6.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}[\ddot{W}_{i,t}Y_{i,t}] = \sum_{g\in\mathcal{G}} \tilde{w}^{g,within}(g)\delta^{WITHIN}(g) + \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Big\{\tilde{w}^{g,post}(g,k)\delta^{MID,PRE}(g,k) + \tilde{w}^{k,post}(g,k)\delta^{MID,POST}(g,k) + \tilde{w}^{long}(g,k)\delta^{POST,PRE}(g,k)\Big\}$$

*Proof.* Notice that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^T \mathbb{E}[\ddot{W}_{i,t}Y_{i,t}] \\
&= \sum_{g\in\mathcal{G}}\Big\{(1-\bar{G}_g)\bar{G}_g \text{Cov}\left(\bar{Y}^{POST(g)}-\bar{Y}^{PRE(g)}, D \,\middle|\, G=g\right)\Big\}p_g \\
&\quad + \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{\mathbb{E}[D|G=g](1-\bar{G}_g)(\bar{G}_g-\bar{G}_k)\Big(\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=k\right]\Big) \\
&\qquad + \mathbb{E}[D|G=k]\bar{G}_k(\bar{G}_g-\bar{G}_k)\Big(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=k\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=g\right]\Big) \\
&\qquad + (\mathbb{E}[D|G=g]-\mathbb{E}[D|G=k])\bar{G}_k(1-\bar{G}_g)\Big(\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{PRE(g)})|G=k\right]\Big)\Bigg\}p_k p_g \\
&= \sum_{g\in\mathcal{G}}\left\{\text{Var}(D|G=g)(1-\bar{G}_g)\bar{G}_g\frac{\text{Cov}\left(\bar{Y}^{POST(g)}-\bar{Y}^{PRE(g)}, D \,\middle|\, G=g\right)}{\text{Var}(D|G=g)}\right\}p_g \\
&\quad + \sum_{g\in\mathcal{G}}\sum_{k\in\mathcal{G},k>g}\Bigg\{\mathbb{E}[D|G=g]^2(1-\bar{G}_g)(\bar{G}_g-\bar{G}_k)\left(\frac{\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=g\right]-\mathbb{E}\left[(\bar{Y}^{MID(g,k)}-\bar{Y}^{PRE(g)})|G=k\right]}{\mathbb{E}[D|G=g]}\right) \\
&\qquad + \mathbb{E}[D|G=k]^2\bar{G}_k(\bar{G}_g-\bar{G}_k)\left(\frac{\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=k\right]-\mathbb{E}\left[(\bar{Y}^{POST(k)}-\bar{Y}^{MID(g,k)})|G=g\right]}{\mathbb{E}[D|G=k]}\right)
\end{aligned}$$

[3] To make this clear, additionally notice that $p_k p_g = (p_g+p_k)^2 p_{g|\{g,k\}}(1-p_{g|\{g,k\}})$ which holds by multiplying and dividing both $p_k$ and $p_g$ by $(p_g+p_k)$ and by the definition of $p_{g|\{g,k\}}$. This expression completely aligns these weights with $w^{g,within}(g)$, $w^{g,post}(g,k)$, $w^{k,post}(g,k)$, and $w^{long}(g,k)$ up to dividing by $T^{-1}\sum_{t=1}^T \mathbb{E}[\ddot{W}_{i,t}^2]$.

$$+ \left(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k]\right)^2 \bar{G}_k(1-\bar{G}_g) \left( \frac{\mathbb{E}\left[ (\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)})|G=g \right] - \mathbb{E}\left[ (\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)})|G=k \right]}{(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])} \right) \Bigg\} p_k p_g$$

$$= \sum_{g \in \mathcal{G}} \tilde{w}^{g,within}(g) \delta^{WITHIN}(g)$$

$$+ \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k>g} \left\{ \tilde{w}^{g,post}(g,k) \delta^{MID,PRE}(g,k) + \tilde{w}^{k,post}(g,k) \delta^{MID,POST}(g,k) + \tilde{w}^{long}(g,k) \delta^{POST,PRE}(g,k) \right\}$$

where the first equality holds from plugging the results of Lemmas S3 and S5 into Equations (S20) and (S21); the second equality holds by multiplying and dividing the first term by $\mathrm{Var}(D|G=g)$, the second term by $\mathbb{E}[D|G=g]$, the third term by $\mathbb{E}[D|G=k]$, and the last term by $(\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k])$; and the third equality holds by the definitions of $\tilde{w}^{g,within}(g)$, $\tilde{w}^{g,post}(g,k)$, $\tilde{w}^{k,post}(g,k)$, $\tilde{w}^{long}(g,k)$, $\delta^{WITHIN}(g)$, $\delta^{MID,PRE}(g,k)$, $\delta^{MID,POST}(g,k)$, $\delta^{POST,PRE}(g,k)$. □

**Lemma S7.** *Under Assumptions 1-MP, 2-MP(a), and 3-MP,*

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\ddot{W}_{i,t}^2] = \sum_{g \in \mathcal{G}} \tilde{w}^{g,within}(g) + \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k>g} \left\{ \tilde{w}^{g,post}(g,k) + \tilde{w}^{k,post}(g,k) + \tilde{w}^{long}(g,k) \right\}$$

*Proof.* To start with, notice that $\mathbb{E}[\ddot{W}_{i,t}^2] = \mathbb{E}[\ddot{W}_{i,t} W_{i,t}]$. Then, we can apply the arguments of Lemmas S2 to S6 but with $W_{i,t}$ replacing $Y_{i,t}$. This implies that

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\ddot{W}_{i,t}^2]$$

$$= \sum_{g \in \mathcal{G}} \tilde{w}^{g,within}(g) \frac{\mathrm{Cov}(\bar{W}^{POST(g)} - \bar{W}^{PRE(g)}, D|G=g)}{\mathrm{Var}(D|G=g)}$$

$$+ \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k>g} \Bigg\{ \tilde{w}^{g,post}(g,k) \frac{\mathbb{E}\left[ (\bar{W}^{MID(g,k)} - \bar{W}^{PRE(g)})|G=g \right] - \mathbb{E}\left[ (\bar{W}^{MID(g,k)} - \bar{W}^{PRE(g)})|G=k \right]}{\mathbb{E}[D|G=g]}$$

$$+ \tilde{w}^{k,post}(g,k) \frac{\mathbb{E}\left[ (\bar{W}^{POST(k)} - \bar{W}^{MID(g,k)})|G=k \right] - \mathbb{E}\left[ (\bar{W}^{POST(k)} - \bar{W}^{MID(g,k)})|G=g \right]}{\mathbb{E}[D|G=k]}$$

$$+ \tilde{w}^{long}(g,k) \frac{\mathbb{E}\left[ (\bar{W}^{POST(k)} - \bar{W}^{PRE(g)})|G=g \right] - \mathbb{E}\left[ (\bar{W}^{POST(k)} - \bar{W}^{PRE(g)})|G=k \right]}{\mathbb{E}[D|G=g] - \mathbb{E}[D|G=k]} \Bigg\}$$

$$= \sum_{g \in \mathcal{G}} \tilde{w}^{g,within}(g) + \sum_{g \in \mathcal{G}} \sum_{k \in \mathcal{G}, k>g} \left\{ \tilde{w}^{g,post}(g,k) + \tilde{w}^{k,post}(g,k) + \tilde{w}^{long}(g,k) \right\}$$

where the last equality holds by noting that $\bar{W} = D$ in post-treatment periods and $\bar{W} = 0$ in pre-treatment periods, and then by canceling terms. □

### Proof of Proposition S1

*Proof.* Proposition S1 immediately holds from applying the result in Lemma S6 in the expression for $\beta^{twfe}$ in Equation (S12). That the weights are all positive holds immediately by their definitions. That they sum to one holds by the definitions of the weights and is an immediate implication of Lemma S7. □

Next, we move to proving Theorem S3. To do this, we provide expressions for each of the comparisons that show up in Proposition S1 in terms of derivatives of paths of outcomes. These results invoke Assumption 2-MP(b) and, therefore, use that the treatment is actually continuous, but they do not invoke any parallel trends assumptions. That said, it would be straightforward to adapt these results to the case with a discrete multi-valued treatment along the lines of the baseline two-period case considered in the main text.

It is also useful to note that

$$\frac{\partial \pi_D^{POST(\tilde{k}),PRE(\tilde{g})}(g,d)}{\partial d} = \frac{\partial \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right) \middle| G=g, D=d\right]}{\partial d},$$
$$\frac{\partial \pi_D^{MID(\tilde{g},\tilde{k}),PRE(\tilde{g})}(g,d)}{\partial d} = \frac{\partial \mathbb{E}\left[\left(\bar{Y}^{MID(\tilde{g},\tilde{k})} - \bar{Y}^{PRE(\tilde{g})}\right) \middle| G=g, D=d\right]}{\partial d},$$
$$\frac{\partial \pi_D^{POST(\tilde{k}),MID(\tilde{g},\tilde{k})}(g,d)}{\partial d} = \frac{\partial \mathbb{E}\left[\left(\bar{Y}^{POST(\tilde{k})} - \bar{Y}^{MID(\tilde{g},\tilde{k})}\right) \middle| G=g, D=d\right]}{\partial d},$$

which holds because the second parts of each $\pi_D$ term do not vary with the dose.

Next, we consider a result for the numerator (which is the main term) of $\delta^{WITHIN}(g)$ in Equation (S13).

**Lemma S8.** *Under Assumptions 1-MP, 2-MP, and 3-MP,*

$$\begin{aligned}&\text{Cov}\left(\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}, D \middle| G=g\right)\\ &= \int_{d_L}^{d_U} \Big(\mathbb{E}[D|G=g, D\geq l] - \mathbb{E}[D|G=g]\Big)\mathbb{P}(D\geq l|G=g)\frac{\partial \mathbb{E}[\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}|G=g, D=l]}{\partial l}\, dl\end{aligned}$$

*Proof.* First, notice that

$$\text{Cov}\left(\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}, D \middle| G=g\right) = \mathbb{E}\big[\big(\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}\big)(D - \mathbb{E}[D|G=g])|G=g\big]$$

Then, the proof follows essentially the same arguments as in Theorem 3.4(a) in the main text with $\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}$ replacing $\Delta Y$ and the other arguments relating to the distribution of the dose holding conditional on being in group $g$. The second term, involving $d_L$, in Theorem 3.4(a) does not show up here as, by construction, there are no untreated units in group $g$. □

Lemma S8 says that part of $\delta^{WITHIN}(g)$ in the TWFE regression estimator comes from a weighted average of $\frac{\partial \mathbb{E}[\bar{Y}^{POST(g)} - \bar{Y}^{PRE(g)}|G=g,D=d]}{\partial d}$.

Next, we consider the numerator (which is the main term) in the expression for $\delta^{MID,PRE}(g,k)$ in Equation (S14). This term is quite similar to the baseline two-period case considered in Theorem 3.4(a) because units in group $k$ have not been treated yet.

**Lemma S9.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for $k > g$,*

$$\begin{aligned}&\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \middle| G=g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \middle| G=k\right]\\ &= \int_{d_L}^{d_U} \mathbb{P}(D\geq l|G=g)\frac{\partial \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|G=g, D=l]}{\partial l}\, dl\\ &\quad + d_L\frac{\mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|G=g, D=d_L] - \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|D=0]}{d_L}\\ &\quad - d_L\frac{\mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|G=k] - \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|D=0]}{d_L}\end{aligned}$$

*Proof.* To start with, notice that

$$\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = k\right]$$
$$= \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right]$$
$$- \left(\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right]\right)$$
$$= \int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = g) \frac{\partial \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | G = g, D = l]}{\partial l} \, dl$$
$$+ d_L \frac{\mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | G = g, D = d_L] - \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | D = 0]}{d_L}$$
$$- d_L \frac{\mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | G = k] - \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | D = 0]}{d_L}$$

where the first equality holds by adding and subtracting $\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \big| D = 0\right]$. For the second equality, notice that

$$\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right]$$
$$= \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g, D = d_L\right]$$
$$+ \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g, D = d_L\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right]$$

Moreover,

$$\mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g, D = d_L\right]$$
$$= \int_{d_L}^{d_U} \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g, D = d\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g, D = d_L\right] dF_{D|G}(d|g)$$
$$= \int_{d_L}^{d_U} \int_{d_L}^{d_U} \mathbf{1}\{l \leq d\} \frac{\partial \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) | G = g, D = l\right]}{\partial l} \, dl \, dF_{D|G}(d|g)$$
$$= \int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = g) \frac{\partial \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} | G = g, D = l]}{\partial l} \, dl$$

where the first equality holds by the law of iterated expectations, the second equality holds by the fundamental theorem of calculus, and the last equality holds by changing the order of integration and simplifying.

Combining the above expressions implies the result. □

Next, we consider the numerator (which is the main term) of $\delta^{POST,MID}(g,k)$ in Equation (S15), which comes from comparing paths of outcomes for newly treated groups relative to already-treated groups.

**Lemma S10.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for $k > g$,*

$$\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = g\right]$$
$$= \int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = k) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)} | G = k, D = l]}{\partial l} \, dl$$
$$+ d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)} | G = k, D = d_L] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)} | D = 0]}{d_L}$$

$$
- \Bigg\{ \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = g] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|D = 0]
$$

$$
- \Big( \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|G = g] - \mathbb{E}[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}|D = 0] \Big) \Bigg\}
$$

*Proof.* Notice that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = g\right] \\
&= \left( \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| D = 0\right] \right) \\
&\quad - \left( \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) | D = 0\right] \right) \\
&= \left( \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right) \Big| D = 0\right] \right) \qquad \text{(S23)} \\
&\quad - \Bigg\{ \left( \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \right) \\
&\quad - \left( \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \right) \Bigg\} \\
&= \int_{d_L}^{d_U} \mathbb{P}(D \geq l|G = k) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}|G = k, D = l]}{\partial l} \, dl \qquad \text{(S24)} \\
&\quad + d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{MID}(g,k)|G = k, D = d_L] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right)|D = 0\right]}{d_L} \\
&- \Bigg\{ \left( \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \right) \\
&\quad - \left( \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \right) \Bigg\}
\end{aligned}
$$

where the first equality holds by adding and subtracting $\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{MID(g,k)}\right)|D = 0\right]$, the second equality holds by adding and subtracting both $\mathbb{E}\left[\bar{Y}^{PRE(g)}|G = g\right]$ and $\mathbb{E}\left[\bar{Y}^{PRE(g)}|D = 0\right]$, and the last equality holds by applying the same sort of arguments as in the proof of Lemma S9. □

Finally, we consider the numerator (which is the main term) of $\delta^{POST,PRE}(g,k)$ in Equation (S16).

**Lemma S11.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for $k > g$,*

$$
\begin{aligned}
&\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right)|G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right)|G = k\right] \\
&= \int_{d_L}^{d_U} \left(\mathbb{P}(D \geq l|G = g) - \mathbb{P}(D \geq l|G = k)\right) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = g, D = l]}{\partial l} \, dl \\
&- \Bigg\{ \int_{d_L}^{d_U} \mathbb{P}(D \geq l|G = k) \left( \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = k, D = l]}{\partial l} - \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = g, D = l]}{\partial l} \right) dl \\
&\quad + d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = k, D = d_L] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|D = 0]}{d_L} \\
&\quad - d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|G = g, D = d_L] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}|D = 0]}{d_L} \Bigg\}.
\end{aligned}
$$

*Proof.* First, by adding and subtracting terms

$$\begin{aligned}
&\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = k\right] \\
&\quad = \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \\
&\qquad - \left(\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right]\right).
\end{aligned}$$

Then, using similar arguments as in Lemma S9 above, one can show that

$$\begin{aligned}
&\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = g\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \\
&\quad = \int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = g) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | G = g, D = l]}{\partial l}\, dl \\
&\qquad + d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | G = g, D = d_L] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | D = 0]}{d_L}
\end{aligned}$$

and that

$$\begin{aligned}
&\mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| G = k\right] - \mathbb{E}\left[\left(\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)}\right) \Big| D = 0\right] \\
&\quad = \int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = k) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | G = k, D = l]}{\partial l}\, dl \\
&\qquad + d_L \frac{\mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | G = k, D = d_L] - \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | D = 0]}{d_L}
\end{aligned}$$

Then, the result holds by adding and subtracting $\int_{d_L}^{d_U} \mathbb{P}(D \geq l | G = k) \frac{\partial \mathbb{E}[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} | G = g, D = l]}{\partial l}\, dl$ and combining terms. □

**Proof of Part (1) of Theorem S3**

*Proof.* Starting from the result in Proposition S1, the expression for $\delta^{WITHIN}(g)$ comes from its definition, the result in Lemma S8, and the definition of the weights $w_1^{within}(g, l)$. The expression for $\delta^{MID,PRE}(g, k)$ comes from its definition, the result in Lemma S9, and the definitions of $w_1(g, l)$ and $w_0(g)$. The expression for $\delta^{POST,MID}(g, k)$ comes from combining its definition with the result in Lemma S10, and the definitions of $w_1(k, l)$ and $w_0(k)$. Finally, the expression for $\delta^{POST,PRE}(g, k)$ comes from its definition, the result in Lemma S11, and the definitions of $w_1^{across}(g, k, l)$, $\tilde{w}_1^{across}(g, k, l)$, and $\tilde{w}_0^{across}(g, k)$.

That $w_1^{within}(g, d) \geq 0$, $w_1(g, d) \geq 0$, $w_0(g) \geq 0$ for all $g \in \mathcal{G}$ and $d \in \mathcal{D}_+^c$ all hold immediately from the definitions of the weights. That $\int_{d_L}^{d_U} w_1^{within}(g, l)\, dl = 1$, $\int_{d_L}^{d_U} w_1(g, l)\, dl + w_0(g) = 1$, and $\int_{d_L}^{d_U} w_1^{across}(g, k, l)\, dl = 1$ hold from the same sorts of arguments used to show that the weights integrate to 1 in the proof of Theorem 3.4(a). □

Notice that none of the previous results have invoked any sort of parallel trends assumption. Next, we push forward the previous results once a researcher invokes parallel trends assumptions; in Theorem S3, we consider the case where the researcher invoked Assumption SPT-MP, but here we handle both that assumption and Assumption PT-MP. To further understand this, for $1 \leq t_1 < t_2 \leq T$

define

$$\bar{Y}_i^{(t_1,t_2)}(g,d) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} Y_{i,t}(g,t,d)$$

which averages potential outcomes from time periods $t_1$ to $t_2$ for unit $i$ if they were in group $g$ and experienced dose $d$. Note that $\bar{Y}_i^{(t_1,t_2)} = \bar{Y}_i^{(t_1,t_2)}(G_i, D_i)$. Next, for $t_1 \leq t_2$, define

$$\overline{ATT}^{(t_1,t_2)}(g,d|g,d) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} ATT(g,t,d|g,d) \tag{S25}$$

which is the average treatment effect experienced by units in group $g$ who experienced dose $d$ averaged across periods from $t_1$ to $t_2$. Likewise, define

$$\overline{ATT}^{(t_1,t_2)}(g,d) = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} ATT(g,t,d) \tag{S26}$$

which is the average treatment effect of dose $d$ among all units in group $g$ averaged across periods from $t_1$ to $t_2$. An alternative expression for $\overline{ATT}^{(t_1,t_2)}(g,d|g,d)$ is given by

$$\overline{ATT}^{(t_1,t_2)}(g,d|g,d) = \mathbb{E}\left[\bar{Y}^{(t_1,t_2)}(g,d) - \bar{Y}^{(t_1,t_2)}(0)|G = g, D = d\right]$$

which holds by the definition of $ATT(g,t,d|g,d)$ and changing the order of the expectation and the average over time periods; here, $\mathbb{E}[\bar{Y}^{(t_1,t_2)}(0)|G = g, D = d]$ is the average outcome that units in group $g$ that experienced dose $d$ would have experienced if they had not participated in the treatment between time periods $t_1$ and $t_2$. Similarly, for $\overline{ATT}^{(t_1,t_2)}(g,d)$,

$$\overline{ATT}^{(t_1,t_2)}(g,d) = \mathbb{E}\left[\bar{Y}^{(t_1,t_2)}(g,d) - \bar{Y}^{(t_1,t_2)}(0)|G = g\right]$$

In addition, define

$$\overline{ACRT}^{(t_1,t_2)}(g,d|g,d) = \frac{\partial \overline{ATT}^{(t_1,t_2)}(g,l|g,d)}{\partial l}\Big|_{l=d} \quad \text{and} \quad \overline{ACRT}^{(t_1,t_2)}(g,d) = \frac{\partial \overline{ATT}^{(t_1,t_2)}(g,d)}{\partial d} \tag{S27}$$

which are the average causal response to a marginal increase in the dose among units in group $g$ conditional on having experienced dose $d$ (for $\overline{ACRT}(g,d|g,d)$) and the average causal response to a marginal increase in the dose among all units in group $g$.

The next result connects derivatives of conditional expectations to $ACRT(g,t,d|g,d)$ and $ACRT(g,t,d)$ parameters under parallel trends assumptions. This is similar to Theorems 3.2, 3.3, and C.1 in the main text and to Theorem S2 above.

**Lemma S12.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for $1 \leq t_1 \leq t_2 < g \leq t_3 \leq t_4 \leq T$ (i.e., $t_1$ and $t_2$ are pre-treatment periods for group $g$, and $t_3$ and $t_4$ are post-treatment periods for group $g$), and for $d \in \mathcal{D}_+^c$,*

*(1) If, in addition, Assumption PT-MP holds, then*

$$\frac{\partial \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)}|G = g, D = d\right]}{\partial d} = \overline{ACRT}^{(t_3,t_4)}(g,d|g,d) + \underbrace{\frac{\partial \overline{ATT}^{(t_3,t_4)}(g,d|g,l)}{\partial l}\Big|_{l=d}}_{\text{selection bias}}$$

*(2) If, in addition, Assumption SPT-MP holds, then*

$$\frac{\partial \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} | G = g, D = d\right]}{\partial d} = \overline{ACRT}^{(t_3,t_4)}(g, d)$$

*Proof.* For part (1), notice that, for $1 \leq t_1 \leq t_2 < g \leq t_3 \leq t_4 \leq T$ (i.e., for group $g$, $t_1$ and $t_2$ are pre-treatment time periods while $t_3$ and $t_4$ are post-treatment time periods), we can write

$$\begin{aligned}
\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \middle| G = g, D = d\right] &= \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g, D = d\right] \\
&= \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d) - \bar{Y}^{(t_3,t_4)}(0) \middle| G = g, D = d\right] \\
&\quad - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g, D = d\right] \\
&= \overline{ATT}^{(t_3,t_4)}(g, d|g, d) \\
&\quad - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g, D = d\right]
\end{aligned}$$

where the first equality holds by writing observed outcomes in terms of their corresponding potential outcomes, the second equality holds by adding and subtracting $\mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) \middle| G = g, D = d\right]$, and the last equality holds by the definition of $\overline{ATT}^{(t_3,t_4)}(g, d|g, d)$.

This equation looks very similar to DiD-type equations in simpler cases, such as when there are two periods and two groups. The left-hand side is immediately identified. The right-hand side involves a causal effect parameter of interest and an unobserved path of untreated potential outcomes that would typically be handled using a parallel trends assumption.

In particular, under Assumption PT-MP,

$$\mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g, D = d\right] = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| D = 0\right]$$

which, importantly, does not vary across $d$ or $g$. This suggests that, under Assumption PT-MP,

$$\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \middle| G = g, D = d\right] = \overline{ATT}^{(t_3,t_4)}(g, d|g, d) - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| D = 0\right]$$

Taking derivatives of both sides of the previous equation with respect to $d$ implies the result.

For part (2), notice that,

$$\begin{aligned}
\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \middle| G = g, D = d\right] &= \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g, D = d\right] \\
&= \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g\right] \\
&= \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d) - \bar{Y}^{(t_3,t_4)}(0) \middle| G = g\right] \\
&\quad + \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| G = g\right] \\
&= \overline{ATT}^{(t_3,t_4)}(g, d) + \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \middle| D = 0\right]
\end{aligned}$$

where the first equality holds by writing observed outcomes in terms of their corresponding potential outcomes, the second equality holds by Assumption SPT-MP, the third equality holds by adding and subtracting $\mathbb{E}[\bar{Y}^{(t_3,t_4)}(0)|G = g]$, and the last equality holds by the definition of $\overline{ATT}^{(t_3,t_4)}(g, d)$ and by Assumption SPT-MP. Taking derivatives of both sides implies the result for part (2). □

The result in Lemma S12 says that, under Assumption PT-MP, the derivative of the path of outcomes (averaged over some post-treatment periods) relative to some pre-treatment periods corresponds to averaging $ACRT(g,t,d|g,d)$ across post-treatment time periods plus the derivative of an averaged selection bias-type across some post-treatment time periods for group $g$. Similarly, under Assumption SPT-MP, the derivative of the path of average outcomes in some post-treatment periods relative to average outcomes in some pre-treatment periods corresponds to an average of $ACRT(g,t,d)$ across the same post-treatment time periods.

**Lemma S13.** *Under Assumptions 1-MP, 2-MP, and 3-MP, and for $1 \leq t_1 \leq t_2 < g \leq t_3 \leq t_4 < k$ (i.e., $t_1$ and $t_2$ are pre-treatment periods for both groups $g$ and $k$, group $g$ is treated before group $k$, and $t_3$ and $t_4$ are post-treatment periods for group $g$ but pre-treatment periods for group $k$),*

*(1) If, in addition, Assumption PT-MP holds, then*

$$d_L \frac{\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)}|G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)}|G=k\right]}{d_L} = d_L \frac{\overline{ATT}^{(t_3,t_4)}(g,d_L|g,d_L)}{d_L}$$

*(2) If, in addition, Assumption SPT-MP holds, then*

$$d_L \frac{\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)}|G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)}|G=k\right]}{d_L} = d_L \frac{\overline{ATT}^{(t_3,t_4)}(g,d_L)}{d_L}$$

*Proof.* For part (1), notice that

$$\begin{aligned}
&\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \,\middle|\, G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \,\middle|\, G=k\right] \\
&\quad = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d_L) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=k\right] \\
&\quad = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d_L) - \bar{Y}^{(t_3,t_4)}(0) \,\middle|\, G=g, D=d_L\right] \\
&\qquad\quad + \left\{\mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=k\right]\right\} \\
&\quad = \overline{ATT}^{(t_3,t_4)}(g,d_L|g,d_L)
\end{aligned}$$

where the first equality holds by writing observed outcomes in terms of their corresponding potential outcomes, the second equality holds by adding and subtracting $\mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0)|G=g, D=d_L\right]$, and the last equality holds by the definition of $\overline{ATT}^{(t_3,t_4)}(g,d_L|g,d_L)$ and because the difference between the two terms involving paths of untreated potential outcomes on the second line of the previous equality is equal to 0 under Assumption PT-MP. Then, the result holds by multiplying and dividing by $d_L$.

For part (2),

$$\begin{aligned}
&\mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \,\middle|\, G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)} - \bar{Y}^{(t_1,t_2)} \,\middle|\, G=k\right] \\
&\quad = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d_L) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=g, D=d_L\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=k\right] \\
&\quad = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d_L) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=g\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=k\right] \\
&\quad = \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(g,d_L) - \bar{Y}^{(t_3,t_4)}(0) \,\middle|\, G=g\right] \\
&\qquad\quad + \left\{\mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=g\right] - \mathbb{E}\left[\bar{Y}^{(t_3,t_4)}(0) - \bar{Y}^{(t_1,t_2)}(0) \,\middle|\, G=k\right]\right\} \\
&\quad = \overline{ATT}^{(t_3,t_4)}(g,d_L)
\end{aligned}$$

where the first equality holds by writing observed outcomes in terms of their corresponding potential outcomes, the second equality holds by Assumption SPT-MP, the third equality holds by adding and subtracting $\mathbb{E}[\bar{Y}^{(t_3,t_4)}(0)|G = g]$, and the last equality holds by Assumption SPT-MP. The result holds by multiplying and dividing by $d_L$. ☐

**Proof of Part (2) of Theorem S3**

*Proof.* The result holds immediately by using the results of Lemmas S12 and S13 in each of the expressions for $\delta^{WITHIN}(g)$, $\delta^{MID,PRE}(g,k)$, $\delta^{POST,MID}(g,k)$, and $\delta^{POST,PRE}(g,k)$ in part (1) of Theorem S3. ☐

**Proof of Proposition S2**

*Proof.* For part (a), we consider the nuisance term involving $\pi^{POST(k),PRE(g)}(g) - \pi^{MID(g,k),PRE(g)}(g)$ in the expression for $\delta^{POST,MID}(g,k)$ in part (2) of Theorem S3. Then, using similar arguments as in Lemma S9 and then under Assumption SPT-MP, it follows that

$$\begin{aligned}\pi^{POST(k),PRE(g)}(g) &= \mathbb{E}\left[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} \,\middle|\, G = g\right] - \mathbb{E}\left[\bar{Y}^{POST(k)} - \bar{Y}^{PRE(g)} \,\middle|\, D = 0\right] \\ &= \int_{d_L}^{d_U} \mathbb{P}(D \geq l|G = g)\overline{ACRT}^{POST(k)}(g,l)\,dl + d_L \frac{\overline{ATT}^{POST(k)}(g,d_L)}{d_L}\end{aligned}$$

and that

$$\begin{aligned}\pi^{MID(g,k),PRE(g)}(g) &= \mathbb{E}\left[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} \,\middle|\, G = g\right] - \mathbb{E}\left[\bar{Y}^{MID(g,k)} - \bar{Y}^{PRE(g)} \,\middle|\, D = 0\right] \\ &= \int_{d_L}^{d_U} \mathbb{P}(D \geq l|G = g)\overline{ACRT}^{MID(g,k)}(g,l)\,dl + d_L \frac{\overline{ATT}^{MID(g,k)}(g,d_L)}{d_L}\end{aligned}$$

Under Assumption S1(a), $ACRT(g,t,d)$ and $ATT(g,t,d_L)$ do not vary over time which implies that, for all $g \in \mathcal{G}$ and $k \in \mathcal{G}$ with $k > g$, $\overline{ACRT}^{POST(k)}(g,l) = \overline{ACRT}^{MID(g,k)}(g,l)$ for all $l \in \mathcal{D}_+^c$ and $\overline{ATT}^{POST(k)}(g,d_L) = \overline{ATT}^{MID(g,k)}(g,d_L)$. This implies that $\pi^{POST(k),PRE(g)}(g) = \pi^{MID(g,k),PRE(g)}(g)$ which implies the result for part (a).

For part (b), we consider the two nuisance terms in the expression for $\delta^{POST,PRE}(g,k)$ in part (2) of Theorem S3. For the first one, notice that, under Assumption SPT-MP,

$$\begin{aligned}\frac{\partial \pi_D^{POST(k),PRE(g)}(k,l)}{\partial l} - \frac{\partial \pi_D^{POST(k),PRE(g)}(g,l)}{\partial l} &= \overline{ACRT}^{POST(k)}(k,l) - \overline{ACRT}^{POST(k)}(g,l) \\ &= 0\end{aligned}$$

for $l \in \mathcal{D}_+^c$ and where the second equality holds by Assumption S1(b) (which implies that, for a particular time period, $ACRT(g,t,d)$ does not vary across groups).

For the second nuisance term, the same sort of arguments imply that

$$\begin{aligned}\frac{\pi_D^{POST(k),PRE(g)}(k,d_L) - \pi_D^{POST(k),PRE(g)}(g,d_L)}{d_L} &= \frac{\overline{ATT}^{POST(k)}(k,d_L) - \overline{ATT}^{POST(k)}(g,d_L)}{d_L} \\ &= 0\end{aligned}$$

under Assumption S1(b).

Finally, for part (c), under Assumption S1(a), (b), and (c), $ACRT(g,t,d)$ does not vary across groups, time periods, or dose; since this does not vary, we denote it by $ACRT$ for the remainder of the proof. Moreover, from Theorem S3, we have that $\int_{d_L}^{d_U} w_1^{within}(g,l)\,dl = 1$, $\int_{d_L}^{d_U} w_1(g,l)\,dl + w_0(g) = 1$, and that $\int_{d_L}^{d_U} w_1^{across}(g,k,l) = 1$. From the first two parts of the current result, we also have that the nuisance paths of outcomes in $\delta^{POST,MID}(g,k)$ and $\delta^{POST,PRE}(g,k)$ are both equal to 0 under Assumption S1(a) and (b). This implies that, under the conditions for part (c), $\delta^{WITHIN}(g) = \delta^{MID,PRE}(g,k) = \delta^{POST,MID}(g,k) = \delta^{POST,PRE}(g,k) = ACRT$. Finally, from Proposition S1, we have that $\beta^{twfe}$ is a weighted average of $\delta^{WITHIN}(g)$, $\delta^{MID,PRE}(g,k)$, $\delta^{POST,MID}(g,k)$, and $\delta^{POST,PRE}(g,k)$. That these are all equal to each other implies that $\beta^{twfe} = ACRT = ACRT^{\texttt{glob}}$. □

# SD Additional Theoretical Results

This appendix provides (and proves) a number of additional results that were referred to in the main text.

## SD.1 No Untreated Units

This section considers the causal interpretation of comparisons of paths of outcomes across dose groups in settings with no untreated units under different versions of the parallel trends assumption.

**Proposition S3.** *Under Assumptions 1, 2, 3, and PT,*[4] *and for* $h, l \in \mathcal{D}_+$,

$$\mathbb{E}[\Delta Y|D=h] - \mathbb{E}[\Delta Y|D=l] = ATT(h|h) - ATT(l|l)$$

*Proof.* Notice that

$$\begin{aligned}
\mathbb{E}[\Delta Y|D=h] - \mathbb{E}[\Delta Y|D=l] &= \mathbb{E}[Y_{t=2}(h) - Y_{t=1}(0)|D=h] - \mathbb{E}[Y_{t=2}(l) - Y_{t=1}(0)|D=l] \\
&= \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(0)|D=h] - \mathbb{E}[Y_{t=2}(l) - Y_{t=2}(0)|D=l] \\
&\quad + \Big(\mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)|D=h] - \mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)|D=l]\Big) \\
&= ATT(h|h) - ATT(l|l)
\end{aligned}$$

where the first equality holds by plugging in potential outcomes for observed outcomes, the second equality holds by adding and subtracting $\mathbb{E}[Y_{t=2}(0)|D=h]$ and $\mathbb{E}[Y_{t=2}(0)|D=l]$, and the last equality holds by the definition of $ATT(d|d)$ and by Assumption PT. □

The result in Proposition S3 is the same as in Theorem 3.2(b), though the proof technique is different here, as there does not exist an untreated comparison group in the setting considered here.

[4] To be fully precise, Assumption 2 needs to be modified here to allow for no untreated units. Likewise, the parallel trends assumption in Assumption PT does not immediately apply to this setting because $\mathbb{P}(D=0)=0$ here. Instead, by parallel trends, we mean that $\mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)|D=d] = \mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)]$ which says that the path of untreated potential outcomes is the same across all dose groups. We do not state this as a separate assumption partly for brevity but also because, in a setting where $\mathbb{P}(D=0) > 0$, the condition here is simply an alternative way to write Assumption PT.

Next, we provide an analogous result under strong parallel trends.

**Proposition S4.** *Under Assumptions 1, 2, 3, and SPT, and for* $h, l \in \mathcal{D}_+$*,*

$$\mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l] = ATT(h) - ATT(l)$$

*Proof.* Notice that

$$\begin{aligned}
\mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l] &= \mathbb{E}[Y_{t=2}(h) - Y_{t=1}(0)|D = h] - \mathbb{E}[Y_{t=2}(l) - Y_{t=1}(0)|D = l] \\
&= \mathbb{E}[Y_{t=2}(h) - Y_{t=1}(0)|D > 0] - \mathbb{E}[Y_{t=2}(l) - Y_{t=1}(0)|D > 0] \\
&= \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(0)|D > 0] - \mathbb{E}[Y_{t=2}(l) - Y_{t=2}(0)|D > 0] \\
&= ATT(h) - ATT(l)
\end{aligned}$$

where the first equality holds by replacing observed outcomes with corresponding potential outcomes, the second equality holds by Assumption SPT, the third equality holds by canceling the $\mathbb{E}[Y_{t=1}(0)|D > 0]$ terms from the previous line and by adding and subtracting $\mathbb{E}[Y_{t=2}(0)|D > 0]$, and the last equality holds by the definition of $ATT(d)$. □

## SD.2 Additional TWFE Decomposition Results

This section provides some extensions and additional details related to the TWFE decompositions discussed in Section 3.3 in the main text.

### Additional Results for TWFE Levels Decomposition

This first part of this section derives the expression for $\beta^{twfe}$ in Equation (3.1) in the main text which relates $\beta^{twfe}$ to a weighted average of "more treated" units (units that experienced a dose larger than $\mathbb{E}[D]$) relative to "less treated" units (units that were untreated or experienced a dose smaller than $\mathbb{E}[D]$) scaled by a weighted average of the difference in treatment experienced by these two groups. Recalling that Theorem 3.4(b) in the main text showed that the "weights" integrated to 0, the second part of this section integrates separately the positive and negative parts of those weights (which are separated on the basis of whether or not $d$ is greater than the mean dose $\mathbb{E}[D]$). The takeaway is that the positive weights do not integrate to 1 (nor do the negative weights integrate to $-1$), but rather they integrate to the reciprocal of the weighted distance between the effective treated and effective comparison group discussed in the main text. This provides an explicit connection between the levels decomposition in Theorem 3.4 and the alternative expression for $\beta^{twfe}$ provided in Equation (3.1) in the main text.

**Corollary S1.** *Under Assumptions 1, 2, 3, and 4(a),*

$$\beta^{twfe} = \frac{\mathbb{E}\Big[w_1^{bin}(D)\Delta Y\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)\Delta Y\Big|D \leq \mathbb{E}[D]\Big]}{\mathbb{E}\Big[w_1^{bin}(D)D\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)D\Big|D \leq \mathbb{E}[D]\Big]}. \tag{S28}$$

*If, in addition, Assumption PT also holds, then*

$$\beta^{twfe} = \frac{\mathbb{E}\Big[w_1^{bin}(D)ATT(D|D)\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)ATT(D|D)\Big|D \leq \mathbb{E}[D]\Big]}{\mathbb{E}\Big[w_1^{bin}(D)D\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)D\Big|D \leq \mathbb{E}[D]\Big]}. \tag{S29}$$

*where*

$$w_1^{bin}(d) := \frac{\big|d - \mathbb{E}[D]\big|}{\mathbb{E}\Big[\big|D - \mathbb{E}[D]\big|\Big|D > \mathbb{E}[D]\Big]}$$

$$w_0^{bin}(d) := \frac{\big|d - \mathbb{E}[D]\big|}{\mathbb{E}\Big[\big|D - \mathbb{E}[D]\big|\Big|D \leq \mathbb{E}[D]\Big]}$$

*which satisfy* $\mathbb{E}\big[w_1^{bin}(D)\big|D > \mathbb{E}[D]\big] = \mathbb{E}\big[w_0^{bin}\big|D \leq \mathbb{E}[D]\big] = 1$.

*Proof.* To start with, recall that

$$\beta^{twfe} = \frac{\mathbb{E}[(D - \mathbb{E}[D])\Delta Y]}{\mathrm{Var}(D)} =: \frac{\beta_{num}}{\beta_{den}}$$

where we consider the numerator and denominator separately below. Next, notice that

$$\begin{aligned} 0 &= \mathbb{E}[(D - \mathbb{E}[D])] \\ &= \mathbb{E}\Big[(D - \mathbb{E}[D])\Big|D \leq \mathbb{E}[D]\Big]\mathbb{P}(D \leq \mathbb{E}[D]) + \mathbb{E}\Big[(D - \mathbb{E}[D])\Big|D > \mathbb{E}[D]\Big]\mathbb{P}(D > \mathbb{E}[D]) \end{aligned}$$

where the second equality holds by the law of iterated expectations. Rearranging the previous expression, we have that

$$\mathbb{E}\Big[|D - \mathbb{E}[D]|\Big|D \leq \mathbb{E}[D]\Big]\mathbb{P}(D \leq \mathbb{E}[D]) = \mathbb{E}\Big[|D - \mathbb{E}[D]|\Big|D > \mathbb{E}[D]\Big]\mathbb{P}(D > \mathbb{E}[D]) =: \delta$$

where the equality uses that the sign of $(D - \mathbb{E}[D])$ is fully determined in both conditional expectations.

Next, similar to above, split the numerator of $\beta^{twfe}$ on the basis of whether or not $D > \mathbb{E}[D]$:

$$\beta_{num} = \mathbb{E}\Big[(D - \mathbb{E}[D])\Delta Y\Big|D > \mathbb{E}[D]\Big]\mathbb{P}(D > \mathbb{E}[D]) + \mathbb{E}\Big[(D - \mathbb{E}[D])\Delta Y\Big|D \leq \mathbb{E}[D]\Big]\mathbb{P}(D \leq \mathbb{E}[D])$$

and now consider,

$$\begin{aligned} \frac{\beta_{num}}{\delta} &= \mathbb{E}\left[\frac{|D - \mathbb{E}[D]|}{\mathbb{E}\big[|D - \mathbb{E}[D]|\big|D > \mathbb{E}[D]\big]}\Delta Y\Big|D > \mathbb{E}[D]\right] - \mathbb{E}\left[\frac{|D - \mathbb{E}[D]|}{\mathbb{E}\big[|D - \mathbb{E}[D]|\big|D \leq \mathbb{E}[D]\big]}\Delta Y\Big|D \leq \mathbb{E}[D]\right] \\ &= \mathbb{E}\Big[w_1^{bin}(D)\Delta Y\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)\Delta Y\Big|D \leq \mathbb{E}[D]\Big] \end{aligned} \tag{S30}$$

which uses the two different expressions for $\delta$ given above. Also, notice that it also immediately follows that $\mathbb{E}[w_1^{bin}(D)|D > \mathbb{E}[D]] = \mathbb{E}[w_0^{bin}|D \leq \mathbb{E}[D]] = 1$. Thus, $\beta_{num}/\delta$ can be thought of as a weighted average of the change in outcomes for units with $D > \mathbb{E}[D]$ relative to a weighted average of the change in outcomes for units with $D \leq \mathbb{E}[D]$, where the weights are larger for units with values of $D$ further away from $\mathbb{E}[D]$.

Similarly, since $\mathrm{Var}(D) = \mathbb{E}[(D - \mathbb{E}[D])D]$, we can apply the same argument to the denominator, and show that

$$\frac{\beta_{den}}{\delta} = \mathbb{E}\Big[w_1^{bin}(D)D\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)D\Big|D \leq \mathbb{E}[D]\Big] \tag{S31}$$

This can be thought of as a weighted average of $D$ for units with $D > \mathbb{E}[D]$ relative to units with $D \leq \mathbb{E}[D]$, or, in other words, the distance between the mean of $D$ for the "effective" treated group relative to the "effective" comparison group given the weighting scheme discussed above. Taking the ratio of Equations S30 and S31 completes the proof for the expression in Equation (S28). That the weights are positive and have mean one follows immediately from their definitions. The result in Equation (S29) holds because

$$\begin{aligned}
\mathbb{E}\Big[w_1^{bin}(D)\Delta Y\Big|D > \mathbb{E}[D]\Big] &= \mathbb{E}\Big[w_1^{bin}(D)\mathbb{E}[\Delta Y|D]\Big|D > \mathbb{E}[D]\Big] \\
&= \mathbb{E}\Big[w_1^{bin}(D)\Big(\mathbb{E}[\Delta Y|D] - \mathbb{E}[\Delta Y|D=0]\Big)\Big|D > \mathbb{E}[D]\Big] + \mathbb{E}[\Delta Y|D=0] \\
&= \mathbb{E}\Big[w_1^{bin}(D)ATT(D|D)\Big|D > \mathbb{E}[D]\Big] + \mathbb{E}[\Delta Y|D=0] \qquad \text{(S32)}
\end{aligned}$$

where the first equality holds by the law of iterated expectations, the second equality holds by adding and subtracting $\mathbb{E}[\Delta Y|D=0]$ and because $\mathbb{E}[\Delta Y|D=0]$ is non-random and $\mathbb{E}\Big[w_1^{bin}(D)\Big|D > \mathbb{E}[D]\Big]$ has mean one, and the last equality holds under Assumption PT. The same sort of argument can be used to show that

$$\mathbb{E}\Big[w_0^{bin}(D)\Delta Y\Big|D \leq \mathbb{E}[D]\Big] = \mathbb{E}\Big[w_0^{bin}(D)ATT(D|D)\Big|D \leq \mathbb{E}[D]\Big] + \mathbb{E}[\Delta Y|D=0] \qquad \text{(S33)}$$

where, by construction, $ATT(0|0) = 0$. Taking the difference between the expressions in Equations (S32) and (S33) and then combining these expressions with the above results for Equation (S28) completes the proof for the expression in Equation (S29).[5] □

**Corollary S2.** *Under Assumptions 1, 2, 3, and 4(a),*

$$-(w_0^{lev} + \int_{d_L}^{\mathbb{E}[D]} w_1^{lev}\, dl) = \int_{\mathbb{E}[D]}^{d_U} w_1^{lev}(l)\, dl = \frac{1}{\mathbb{E}\Big[w_1^{bin}(D)D\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)D\Big|D \leq \mathbb{E}[D]\Big]}$$

*where $w_1^{bin}$ and $w_0^{bin}$ are defined in Corollary S1.*

*Proof.* That $-(w_0^{lev} + \int_{d_L}^{\mathbb{E}[D]} w_1^{lev}(l)\, dl) = \int_{\mathbb{E}[D]}^{d_U} w_1^{lev}(l)\, dl$ follow from Theorem 3.4(b) and linearity of integrals. Therefore, consider

$$\begin{aligned}
\int_{\mathbb{E}[D]}^{d_U} w_1^{lev}(l)\, dl &= \int_{\mathbb{E}[D]}^{d_U} \frac{(l - \mathbb{E}[D])}{\mathrm{Var}(D)} f_D(l)\, dl \\
&= \frac{\mathbb{E}\Big[|D - \mathbb{E}[D]|\Big|D > \mathbb{E}[D]\Big]\mathbb{P}(D > \mathbb{E}[D])}{\mathrm{Var}(D)} \\
&= \frac{\delta}{\beta_{den}} \\
&= \frac{1}{\mathbb{E}\Big[w_1^{bin}(D)D\Big|D > \mathbb{E}[D]\Big] - \mathbb{E}\Big[w_0^{bin}(D)D\Big|D \leq \mathbb{E}[D]\Big]}
\end{aligned}$$

where the first equality holds by the definition of $w_1^{lev}(l)$, the second equality holds by the law of iterated expectations and because $(D - \mathbb{E}[D])$ is positive conditional on $D > \mathbb{E}[D]$, the third equality

[5] Notice that if we were to invoke Assumption SPT, a result analogous to the one in Equation (S29) holds with $ATT(D)$ replacing $ATT(D|D)$.

holds from the expressions for $\delta$ and $\beta_{den}$ in the proof of Corollary S1, and the last equality holds by Equation (S31) above. This completes the proof. $\square$

### Scaled-Levels Decomposition for Fixed Dose

Next, we consider interpreting $\beta^{twfe}$ as $ATT(d)/d$ for some particular fixed value of $d$. This is similar to the scaled-level effects discussed in Section 3.3 in the main text except that we fix $d$ instead of relating $\beta^{twfe}$ to a weighted average of this type of scaled level effect across all values of the dose.

In this section and the next, we define the following weights

$$w^{diff}(d_1, d_2) := \frac{1}{d_2 - d_1}$$
$$w_1^{s,+}(d) := \frac{d - d_L}{d}$$

Also, recall that we defined $m_\Delta(d) = \mathbb{E}[\Delta Y | D = d]$ in the main text—we use this shorthand notation in the results below.

**Proposition S5.** *Under Assumptions 1, 2, 3, 4(a), and PT,*

$$\begin{aligned}\frac{ATT(d|d)}{d} - \beta^{twfe} &= \Big(1 - w_1^{s,+}(d)\Big)\frac{ATT(d_L|d_L)}{d_L}\underbrace{\left(1 - \frac{w_0^{acr}}{(1 - w_1^{s,+}(d))}\right)} \\ &\quad + \int_{d_L}^{d_U} w_1^{s,+}(d) w^{diff}(d, d_L) m'_\Delta(l) \underbrace{\Big(1 - d w_1^{acr}(l)\Big)}\, dl \\ &\quad - \left\{\int_d^{d_U} m'_\Delta(l) w_1^{acr}(l)\, dl\right\}\end{aligned}$$

*where* $m'_\Delta(l) = ACRT(l|l) + \left.\frac{\partial ATT(l|h)}{\partial h}\right|_{h=l}$.

*If Assumption SPT holds instead of Assumption PT, then the same sort of result holds with $ATT(d)$ replacing $ATT(d|d)$ on the LHS of the previous equation and with $m'_\Delta(l) = ACRT(l)$ on the RHS of the previous equation.*

*Proof.* To start with, consider the path of outcomes experienced by dose group $d$ relative to the untreated group scaled by $d$:

$$\begin{aligned}\frac{m_\Delta(d) - m_\Delta(0)}{d} &= \frac{m_\Delta(d) - m_\Delta(d_L)}{d} + \frac{m_\Delta(d_L) - m_\Delta(0)}{d} \\ &= \frac{(d - d_L)}{d}\frac{m_\Delta(d) - m_\Delta(d_L)}{d - d_L} + \frac{d_L}{d}\frac{m_\Delta(d_L) - m_\Delta(0)}{d_L} \\ &= \frac{(d - d_L)}{d}\frac{\int_{d_L}^d m'_\Delta(l)\, dl}{d - d_L} + \frac{d_L}{d}\frac{m_\Delta(d_L) - m_\Delta(0)}{d_L} \\ &= w_1^{s,+}(d)\int_{d_L}^d w^{diff}(d, d_L) m'_\Delta(l)\, dl + \Big(1 - w_1^{s,+}(d)\Big)\frac{m_\Delta(d_L) - m_\Delta(0)}{d_L} \qquad \text{(S34)}\end{aligned}$$

where the first equality holds by adding and subtracting $m_\Delta(d_L)/d$, the second equality holds by multiplying and dividing the first term by $(d - d_L)$ and the second term by $d_L$, the third equality holds by the fundamental theorem of calculus, and the last line holds by the definitions of $w^{diff}$ and

$w_1^{s,+}$. Further, notice that the weights integrate/sum to 1:

$$w_1^{s,+}(d)\int_{d_L}^{d} w^{diff}(d,d_L)\,dl + \Big(1-w_1^{s,+}(d)\Big) = \frac{(d-d_L)}{d}\underbrace{\frac{1}{d-d_L}\int_{d_L}^{d} dl}_{=1} + \frac{d_L}{d} = 1$$

which suggests interpreting $(m_\Delta(d)-m_\Delta(0))/d$ as an average of derivative-type terms. Then, using a similar argument for $\beta^{twfe}$ as the one used in Equation (S37) below and combining it with the expression in Equation (S34), we have that

$$\begin{aligned}\frac{m_\Delta(d)-m_\Delta(0)}{d} - \beta^{twfe} = &\Big(1-w_1^{s,+}(d)\Big)\frac{(m_\Delta(d_L)-m_\Delta(0))}{d_L}\left(1-\frac{w_0^{acr}}{\big(1-w_1^{s,+}(d)\big)}\right)\\ &+\int_{d_L}^{d_U} w_1^{s,+}(d)w^{diff}(d,d_L)m'_\Delta(l)\Big(1-dw_1^{acr}(l)\Big)\,dl\\ &-\left\{\int_{d}^{d_U} m'_\Delta(l)w_1^{acr}(l)\,dl\right\}\end{aligned}$$

As in Theorem 3.1, under Assumption PT, $m_\Delta(d)-m_\Delta(0) = ATT(d|d)$, and, as in Theorem 3.2, $m'_\Delta(l) = ACRT(l|l) + \frac{\partial ATT(l|h)}{\partial h}\Big|_{h=l}$ (notice that this term includes selection bias). Under Assumption SPT, $m_\Delta(d)-m_\Delta(0) = ATT(d)$ and $m'_\Delta(l) = ACRT(l)$. This completes the proof. □

In other words, in general, $\beta^{twfe}$ is not equal to $ATT(d|d)/d$ (under parallel trends) or $ATT(d)/d$ (under strong parallel trends) for two reasons: (i) they put different weights on the same effects (the underlined terms above), and (ii) the value of $\beta^{twfe}$ additionally depends on effects of the treatment for doses greater than $d$ (the third term, in brackets, in the proposition).

### Scaled-2 × 2 Decomposition for Fixed Doses

Finally, we consider interpreting $\beta^{twfe}$ as $\frac{ATT(d_2|d_2)-ATT(d_1|d_1)}{d_2-d_1}$ or $\frac{ATT(d_2)-ATT(d_1)}{d_2-d_1}$ for two fixed doses $d_1$ and $d_2$ that satisfy $d_L < d_1 < d_2 < d_U$. This is similar to the scaled $2\times 2$ effects discussed in Section 3.3 except that here we fix the values of $d_1$ and $d_2$ rather than relating $\beta^{twfe}$ to a weighted average of all possible scaled $2\times 2$ effects.

**Proposition S6.** *Under Assumptions 1, 2, 3, 4(a), and PT and for $d_L < d_1 < d_2 < d_U$,*

$$\begin{aligned}&\frac{ATT(d_2|d_2)-ATT(d_1|d_1)}{d_2-d_1} - \beta^{twfe}\\ &= \int_{d_1}^{d_2} w^{diff}(d_1,d_2)m'_\Delta(l)\Big(1-(d_2-d_1)w_1^{acr}(l)\Big)\,dl\\ &\quad -\left\{\int_{d_L}^{d_1} m'_\Delta(l)w_1^{acr}(l)\,dl + \int_{d_2}^{d_U} m'_\Delta(l)w_1^{acr}(l)\,dl + w_0^{acr}\frac{(m_\Delta(d_L)-m_\Delta(0))}{d_L}\right\}\end{aligned}$$

*where $m'_\Delta(l) = ACRT(l|l) + \frac{\partial ATT(l|h)}{\partial h}\Big|_{h=l}$.*

*If Assumption SPT holds instead of Assumption PT, then the same sort of result holds with $ATT(d_2)-ATT(d_1)$ replacing $ATT(d_2|d_2)-ATT(d_1|d_1)$ on the LHS of the previous equation and with $m'_\Delta(l) = ACRT(l)$ on the RHS of the previous equation.*

*Proof.* To start with, consider the path of outcomes under dose $d_2$ relative to the path of outcomes under dose $d_1$ scaled by $(d_2 - d_1)$, and notice that

$$\frac{m_\Delta(d_2) - m_\Delta(d_1)}{d_2 - d_1} = \int_{d_1}^{d_2} \frac{1}{d_2 - d_1} m'_\Delta(l)\, dl = \int_{d_1}^{d_2} w^{diff}(d_1, d_2) m'_\Delta(l)\, dl \tag{S35}$$

where the first equality holds by the fundamental theorem of calculus, and the second equality by the definition of $w^{diff}$. In addition, notice that the "weights" here integrate to one:

$$\int_{d_1}^{d_2} w^{diff}(d_1, d_2)\, dl = \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} dl = 1$$

Now, move to considering $\beta^{twfe}$. From Equation (S2) in the proof of Theorem 3.4 in the main text, we have that

$$\begin{aligned}\beta^{twfe} =& \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\text{Var}(D)}(m_\Delta(D) - m_\Delta(d_L))\Big| D > 0\right] \mathbb{P}(D > 0) \\ &+ \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\text{Var}(D)}(m_\Delta(d_L) - m_\Delta(0))\Big| D > 0\right] \mathbb{P}(D > 0)\end{aligned}$$

Focusing on the first term in the above expression, and, again, from the proof of Theorem 3.4, we have that

$$\begin{aligned}&\mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\text{Var}(D)}(m_\Delta(D) - m_\Delta(d_L))\Big| D > 0\right] \mathbb{P}(D > 0) \\ &\quad = \int_{d_L}^{d_U} m'_\Delta(l) w_1^{acr}(l)\, dl \\ &\quad = \left\{\int_{d_L}^{d_1} m'_\Delta(l) w_1^{acr}(l)\, dl + \int_{d_1}^{d_2} m'_\Delta(l) w_1^{acr}(l)\, dl + \int_{d_2}^{d_U} m'_\Delta(l) w_1^{acr}(l)\, dl\right\}\end{aligned} \tag{S36}$$

where the second equality just splits the integral into three parts and, as in the main text, $w_1^{acr}(l) = \frac{(\mathbb{E}[D|D \geq l] - \mathbb{E}[D])\mathbb{P}(D \geq l)}{\text{Var}(D)}$. Taking the difference between the expressions in Equations S35 and the representation of $\beta^{twfe}$ above, we have that

$$\begin{aligned}\frac{m_\Delta(d_2) - m_\Delta(d_1)}{d_2 - d_1} - \beta^{twfe} =& \int_{d_1}^{d_2} w^{diff}(d_1, d_2) m'_\Delta(l)\Big(1 - (d_2 - d_1) w_1^{acr}(l)\Big)\, dl \\ &- \left\{\int_{d_L}^{d_1} m'_\Delta(l) w_1^{acr}(l)\, dl + \int_{d_2}^{d_U} m'_\Delta(l) w_1^{acr}(l)\, dl + w_0^{acr} \frac{(m_\Delta(d_L) - m_\Delta(0))}{d_L}\right\}\end{aligned} \tag{S37}$$

where, as in the main text, $w_0^{acr} = \frac{(\mathbb{E}[D|D>0] - \mathbb{E}[D])\mathbb{P}(D>0) d_L}{\text{Var}(D)}$.

As in Theorem 3.2, under Assumption PT, $m_\Delta(d_2) - m_\Delta(d_1) = ATT(d_2|d_2) - ATT(d_1|d_1) = \mathbb{E}[Y_{t=2}(d_2) - Y_{t=2}(d_1)|D = d_2] + \Big(ATT(d_1|d_2) - ATT(d_2|d_2)\Big)$ and $m'_\Delta(l) = ACRT(l|l) + \frac{\partial ATT(l|h)}{\partial h}\Big|_{h=l}$ (notice that both of these expressions also include selection bias). Under Assumption SPT, $m_\Delta(d_2) - m_\Delta(d_1) = ATT(d_2) - ATT(d_1)$ and $m'_\Delta(l) = ACRT(l)$. This completes the proof. □

This shows that, in general, $\beta^{twfe}$ will be different from $\frac{ATT(d_2|d_2) - ATT(d_1|d_1)}{d_2 - d_1}$ (under parallel trends) or $\frac{ATT(d_2) - ATT(d_1)}{d_2 - d_1}$ (under strong parallel trends) due to (i) different weights on underlying derivative terms (i.e., $m'_\Delta(l)$) for values of $l$ between $d_1$ and $d_2$ (this is the underlined term in the expression in the proposition), and (ii) because $\beta^{twfe}$ additionally depends on effects of the treatment for values outside of $[d_1, d_2]$ (this is the second term in curly brackets in the expression in the proposition).

## SD.3 TWFE Decomposition with a Multi-Valued Discrete Treatment

The following theorem provides the discrete analog of Theorem 3.4 from the main text. The weights in the decomposition are the same ones as those used in the main text, which are reported in Table 1 in the main text, with the exception that $f_D(l)$ should be understood as $p_l$ in the discrete case. In this section, we continue to use the notation $m_\Delta(d) = \mathbb{E}[\Delta Y|D = d]$.

**Theorem S4.** *Under Assumptions 1, 2, 3, 4(b), and PT, $\beta^{twfe}$ can be decomposed in the following ways:*

*(a) Causal Response Decomposition:*

$$\beta^{twfe} = \sum_{d_j \in \mathcal{D}_+^{mv}} w_1^{acr}(d_j)(d_j - d_{j-1}) \left\{ ACRT(d_j|d_j) + \underbrace{\frac{\Big(ATT(d_{j-1}|d_j) - ATT(d_{j-1}|d_{j-1})\Big)}{d_j - d_{j-1}}}_{\text{selection bias}} \right\}$$

*where the weights, $w_1^{acr}(d_j)(d_j - d_{j-1})$ are always positive and sum to 1.*

*(b) Levels Decomposition:*

$$\beta^{twfe} = \sum_{d_j \in \mathcal{D}_+^{mv}} w_1^{lev}(d_j) ATT(d_j|d_j)$$

*where $w_1^{lev}(d_j) \lesseqgtr 0$ for $d_j \lesseqgtr \mathbb{E}[D]$, and $\sum_{d_j \in \mathcal{D}_+^{mv}} w_1^{lev}(d_j) + w_0^{lev} = 0$.*

*(c) Scaled Levels Decomposition:*

$$\beta^{twfe} = \sum_{d_j \in \mathcal{D}_+^{mv}} w^s(d_j) \frac{ATT(d_j|d_j)}{d_j},$$

*where $w^s(d_j) \lesseqgtr 0$ for $d_j \lesseqgtr \mathbb{E}[D]$, and $\sum_{d_j \in \mathcal{D}_+^{mv}} w^s(d_j) = 1$.*

*(d) Scaled $2 \times 2$ Decomposition*

$$\beta^{twfe} = \sum_{l \in \mathcal{D}} \sum_{h \in \mathcal{D}, h > l} w_1^{2\times 2}(l, h) \left\{ \underbrace{\frac{\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h]}{h - l}}_{\text{causal response}} + \underbrace{\frac{\Big(ATT(l|h) - ATT(l|l)\Big)}{h - l}}_{\text{selection bias}} \right\}$$

*where the weights are always positive and sum to 1.*

*If one imposes Assumption SPT instead of Assumption PT, then the selection bias terms from part (a) and part (d) become zero, and the remainder of the decompositions remain true, except one needs to replace $ACRT(d_j|d_j)$ with $ACRT(d_j)$ in part (a), $ATT(d_j|d_j)$ with $ATT(d_j)$ in parts (b) and (c), and $\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h]$ with $\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D > 0]$ in part (d).*

## Proof of Theorem S4

We follow the same proof strategy as for the continuous case in the main text and mainly emphasize the parts of the proof that are different from those in the continuous case. As in the continuous case, our strategy is to provide a mechanical decomposition in terms of $m_\Delta(d) = \mathbb{E}[\Delta Y|D = d]$. Then, given those results, the main results in the theorem hold because, under Assumption PT

- $m_\Delta(d_j) - m_\Delta(0) = ATT(d_j|d_j)$
- $\frac{m_\Delta(d_j)-m_\Delta(d_{j-1})}{d_j-d_{j-1}} = ACRT(d_j|d_j) + \underbrace{\frac{ATT(d_{j-1}|d_j) - ATT(d_{j-1}|d_{j-1})}{d_j - d_{j-1}}}_{\text{selection bias}}$
- For $h, l \in \mathcal{D}_+^{mv}$, $m_\Delta(h) - m_\Delta(l) = ATT(h|h) - ATT(l|l) = \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h] + \underbrace{\Big(ATT(l|h) - ATT(l|l)\Big)}_{\text{selection bias}}$

or, when Assumption SPT holds,

- $m_\Delta(d_j) - m_\Delta(0) = ATT(d_j)$
- $\frac{m_\Delta(d_j)-m_\Delta(d_{j-1})}{d_j-d_{j-1}} = ACRT(d_j)$
- For $h, l \in \mathcal{D}_+^{mv}$, $m_\Delta(h) - m_\Delta(l) = ATT(h) - ATT(l) = \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D > 0]$

## Proof of Theorem S4(a)

*Proof.* Notice that,

$$\begin{aligned}
\beta^{twfe} &= \mathbb{E}\left[\frac{(D - \mathbb{E}[D])}{\text{Var}(D)}(m_\Delta(D) - m_\Delta(0))\right] \\
&= \frac{1}{\text{Var}(D)} \sum_{d \in \mathcal{D}} (d - \mathbb{E}[D])(m_\Delta(d) - m_\Delta(0)) p_d \\
&= \frac{1}{\text{Var}(D)} \sum_{d \in \mathcal{D}} (d - \mathbb{E}[D]) p_d \sum_{d_j \in \mathcal{D}_+^{mv}} \mathbf{1}\{d_j \leq d\}(m_\Delta(d_j) - m_\Delta(d_{j-1})) \\
&= \frac{1}{\text{Var}(D)} \sum_{d_j \in \mathcal{D}_+^{mv}} (m_\Delta(d_j) - m_\Delta(d_{j-1})) \sum_{d \in \mathcal{D}} (d - \mathbb{E}[D]) \mathbf{1}\{d \geq d_j\} p_d \\
&= \sum_{d_j \in \mathcal{D}_+^{mv}} (m_\Delta(d_j) - m_\Delta(d_{j-1})) \frac{(\mathbb{E}[D|D \geq d_j] - \mathbb{E}[D])\mathbb{P}(D \geq d_j)}{\text{Var}(D)} \\
&= \sum_{d_j \in \mathcal{D}_+^{mv}} w_1^{acr}(d_j)(d_j - d_{j-1}) \frac{(m_\Delta(d_j) - m_\Delta(d_{j-1}))}{(d_j - d_{j-1})}
\end{aligned}$$

where the second equality holds by writing the expectation as a summation, the third equality holds by adding and subtracting $m_\Delta(d_j)$ for all $d_j$'s between 0 and $d$, the fourth equality holds by changing the order of the summations, the fifth equality writes the second summation as an expectation, and the last equality holds by the definition of the weights and by multiplying and dividing by $(d_j - d_{j-1})$.

That $w_1^{acr}(d_j)(d_j - d_{j-1}) > 0$ holds immediately since $w_1^{acr}(d_j) \geq 0$ for all $d_j \in \mathcal{D}_+^{mv}$ and $d_j > d_{j-1}$. Further,

$$
\begin{aligned}
&\sum_{d_j \in \mathcal{D}_+^{mv}} w_1^{acr}(d_j)(d_j - d_{j-1}) \\
&= \left( \sum_{d_j \in \mathcal{D}_+^{mv}} \mathbb{E}[\mathbf{1}\{D \geq d_j\} D](d_j - d_{j-1}) - \mathbb{E}[D] \sum_{d_j \in \mathcal{D}_+^{mv}} \mathbb{P}(D \geq d_j)(d_j - d_{j-1}) \right) \Big/ \mathrm{Var}(D) \\
&= (A - B)/\mathrm{Var}(D)
\end{aligned}
$$

We consider each of these terms in turn:

$$
\begin{aligned}
A &= \sum_{d_j \in \mathcal{D}_+^{mv}} \sum_{d_k \in \mathcal{D}} \mathbf{1}\{d_k \geq d_j\} d_k p_{d_k} (d_j - d_{j-1}) \\
&= \sum_{d_k \in \mathcal{D}} p_{d_k} d_k \sum_{d_j \in \mathcal{D}_+^{mv}, d_j \leq d_k} (d_j - d_{j-1}) \\
&= \sum_{d_k \in \mathcal{D}} p_{d_k} d_k (d_k - 0) \\
&= \mathbb{E}[D^2]
\end{aligned}
$$

where the first equality holds by writing the expectation for Term A as a summation, the second equality holds by re-ordering the summations, the third equality holds by canceling all the duplicate $d_j$ terms across summations (and because $d_0 = 0$), and the last equality holds by the definition of $\mathbb{E}[D^2]$.

Next,

$$
\begin{aligned}
B &= \mathbb{E}[D] \sum_{d_j \in \mathcal{D}_+^{mv}} \sum_{d_k \in \mathcal{D}} \mathbf{1}\{d_k \geq d_j\} p_{d_k} (d_j - d_{j-1}) \\
&= \mathbb{E}[D] \sum_{d_k \in \mathcal{D}} p_{d_k} \sum_{d_j \in \mathcal{D}_+^{mv}, d_j \leq d_k} (d_j - d_{j-1}) \\
&= \mathbb{E}[D] \sum_{d_k \in \mathcal{D}} d_k p_{d_k} \\
&= \mathbb{E}[D]^2
\end{aligned}
$$

where the first equality holds by writing the expectation for Term B as a summation, the second equality holds by re-ordering the summations, the third equality holds by canceling all the duplicate $d_j$ terms across summations (and because $d_0 = 0$), and the last equality holds by the definition of $\mathbb{E}[D]$.

This implies that $A - B = \mathrm{Var}(D)$, which implies that the weights sum to 1. □

## Proof of Theorem S4(b)

*Proof.* The proof is analogous to the continuous case in Theorem 3.4(b) in the main text except for replacing the integral with a summation and $f_D(l)$ with $p_l$. Then the result holds by the definition of $w^{lev}$. □

### Proof of Theorem S4(c)

*Proof.* The proof is analogous to the continuous case in Theorem 3.4(c) in the main text except for replacing the integral with a summation and $f_D(l)$ with $p_l$. Then the result holds by the definition of $w^s$. $\square$

### Proof of Theorem S4(d)

*Proof.* Up to Equation (S10) in the main text, the steps of the proof of Theorem 3.4(d) for the continuous case carry over to the discrete case. Under Assumption 4(b),

$$\text{Equation (S10)} = \frac{1}{\text{Var}(D)} \sum_{l \in \mathcal{D}} \sum_{h \in \mathcal{D}, h > l} (h-l)^2 \frac{(m_\Delta(h) - m_\Delta(l))}{(h-l)} p_h p_l$$

which holds immediately from Equation (S10) and then the result holds by the definition of $w_1^{2\times 2}$. That the weights are positive and sum to 1 holds by the same type of argument as used in the continuous case. $\square$

## SE Relaxing Strong Parallel Trends

In this section, we provide more details about the three possible ideas to weaken the strong parallel trends assumption that were discussed in Section 5 in the main text.

### SE.1 Partial Identification

To start with, we consider the case where a researcher only wishes to invoke parallel trends (Assumption PT) but is willing to assume that the sign of the selection bias is known. We focus on the case where there is positive selection bias in the sense that, for dose $d$ and any two dose groups $l$ and $h$ with $l < h$, we have that $ATT(d|l) \leq ATT(d|h)$—this is positive selection bias in that the $ATT$ of any dose is higher for the high dose group, $h$, relative to the low dose group, $l$. The following result shows that, under this sort of condition, we can construct bounds on differences between causal effect parameters at different values of the dose.

**Proposition S7.** *Under Assumptions 1, 2, 3, 4(a), and PT and suppose that for any $d \in \mathcal{D}_+$ and $l < h$, $ATT(d|l) < ATT(d|h)$, then the following results hold*

*(1)* $\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h] \leq \mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l] = ATT(h|h) - ATT(l|l)$

*(2)* $ACRT(d|d) \leq \dfrac{\partial \mathbb{E}[\Delta Y|D = d]}{\partial d}$

*Proof.* For part (1), from Theorem 3.2(a) in the main text, we have that, under Assumption PT,

$$\begin{aligned}\mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l] &= ATT(h|h) - ATT(l|l) \\ &= \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h] + \underbrace{\Big(ATT(l|h) - ATT(l|l)\Big)}_{\geq 0}\end{aligned}$$

$$\geq \mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D = h]$$

where the last inequality holds due to the positive selection bias.

For part (2), from Theorem 3.2(b) in the main text, we have that

$$\frac{\partial \mathbb{E}[\Delta Y|D = d]}{\partial d} = ACRT(d|d) + \underbrace{\left.\frac{\partial ATT(d|l)}{\partial l}\right|_{l=d}}_{\geq 0}$$

$$\geq ACRT(d|d)$$

where the last inequality holds due to the positive selection bias. □

Part (1) of Proposition S7 says that, given positive selection bias, the average causal response of the high dose, $h$, relative to the low dose, $l$, for the high dose group is bounded by comparing the average path of outcomes over time for the high dose group relative to the low dose group. Part (2) says that, under positive selection bias, the $ACRT(d|d)$ is bounded by the derivative of $\mathbb{E}[\Delta Y|D = d]$ with respect to $d$.

## SE.2 Local Strong Parallel Trends

In this section, we consider a local strong parallel trends assumption where, as discussed in the main text, strong parallel trends holds in some sub-region $\mathcal{D}_s \subseteq \mathcal{D}_+$. As discussed in the main text, we focus on identifying a local causal effect parameter given by $\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D \in \mathcal{D}_s]$ for $h, l \in \mathcal{D}_s$. This is the average causal effect of experiencing dose $h$ relative to dose $l$ among all dose groups that experienced a treatment in $\mathcal{D}_s$. We consider the following assumption

**Assumption Local-SPT.** *For all $d \in \mathcal{D}_s \subseteq \mathcal{D}_+$,*

$$\mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|D \in \mathcal{D}_s] = \mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|D = d]$$

This assumption is similar to Assumption SPT from the main text, with the difference being that it holds locally to the sub-region $\mathcal{D}_s$. It is also different in spirit from Assumption PT as it does not require the dose groups in $\mathcal{D}_s$ to be experiencing the same trend in untreated potential outcomes as the untreated group. Next, we show that, for $h, l \in \mathcal{D}_s$, the average causal effect of experiencing dose $h$ relative to dose $l$ across dose groups in $\mathcal{D}_s$, $\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D \in \mathcal{D}_s]$, is identified under this assumption.

**Proposition S8.** *Under Assumptions 1, 2, 3, 4(a), and Local-SPT, and for $h, l \in \mathcal{D}_s$*

$$\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D \in \mathcal{D}_s] = \mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l]$$

*Proof.* For any $h, l \in \mathcal{D}_s$, we have that

$$\begin{aligned}
\mathbb{E}[Y_{t=2}(h) - Y_{t=2}(l)|D \in \mathcal{D}_s] &= \mathbb{E}[Y_{t=2}(h) - Y_{t=1}(0)|D \in \mathcal{D}_s] - \mathbb{E}[Y_{t=2}(l) - Y_{t=1}(0)|D \in \mathcal{D}_s] \\
&= \mathbb{E}[Y_{t=2}(h) - Y_{t=1}(0)|D = h] - \mathbb{E}[Y_{t=2}(l) - Y_{t=1}(0)|D = l] \\
&= \mathbb{E}[\Delta Y|D = h] - \mathbb{E}[\Delta Y|D = l]
\end{aligned}$$

where the first equality holds by adding and subtracting $\mathbb{E}[Y_{t=1}(0)|D \in \mathcal{D}_s]$, the second equality uses Local-SPT, and the last equality holds by replacing potential outcomes with their observed counterparts. ☐

An immediate corollary to the previous result is that a local version of $ACRT(d)$ is also identified: $\frac{\partial \mathbb{E}[Y_{t=2}(d)|D \in \mathcal{D}_s]}{\partial d} = \frac{\partial \mathbb{E}[\Delta Y|D = d]}{\partial d}$ for $d$ in the interior of $\mathcal{D}_s$—notice that there are no selection bias terms in this expression which is due to this being a version of strong parallel trends.

### SE.3 Strong Parallel Trends Conditional-on-Covariates

In this section, we consider a conditional-on-covariates version of strong parallel trends that can be used to recover conditional versions of $ATT(d)$ parameters. We target $ATT_x(d) := \mathbb{E}[Y_{t=2}(d) - Y_{t=2}(0)|X = x, D > 0]$. We consider the following assumption

**Assumption SPT-X.** *For all $d \in \mathcal{D}$,*

$$\mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|X = x, D > 0] = \mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|X = x, D = d]$$

This is a conditional-on-covariates version of strong parallel trends. The following result shows that $ATT_x(d)$ is identified under this condition.

**Proposition S9.** *Under Assumptions 1, 2, 3, 4(a), and SPT-X,*[6]

$$ATT_x(d) = \mathbb{E}[\Delta Y|X = x, D = d] - \mathbb{E}[\Delta Y|X = x, D = 0]$$

*Proof.* For any $d \in \mathcal{D}_+$, we have that

$$\begin{aligned}
ATT_x(d) &= \mathbb{E}[Y_{t=2}(d) - Y_{t=2}(0)|X = x, D > 0] \\
&= \mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|X = x, D > 0] - \mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)|X = x, D > 0] \\
&= \mathbb{E}[Y_{t=2}(d) - Y_{t=1}(0)|X = x, D = d] - \mathbb{E}[Y_{t=2}(0) - Y_{t=1}(0)|X = x, D = 0] \\
&= \mathbb{E}[\Delta Y|X = x, D = d] - \mathbb{E}[\Delta Y|X = x, D = 0],
\end{aligned}$$

where the first equality holds by the definition of $ATT_x(d)$, the second equality holds by adding and subtracting $\mathbb{E}[Y_{t=1}(0)|X = x, D > 0]$, the third equality holds by Assumption SPT-X, and the last equality by replacing potential outcomes with their observed counterparts. ☐

An immediate corollary to the previous result is that the conditional on covariates version of $ACRT(d)$ is also identified. In particular, $ACRT_x(d) := \frac{\partial ATT_x(d)}{\partial d} = \frac{\partial \mathbb{E}[\Delta Y|X = x, D = d]}{\partial d}$. Notice that there is no selection bias term in this expression.

## References


de Chaisemartin, Clement and Xavier D'Haultfoeuille (2020). "Two-way fixed effects estimators with heterogeneous treatment effects". *American Economic Review* 110.9, pp. 2964–2996.


[6] Assumptions 1, 2, and 4(a) need to be slightly modified so that $X_i$ is included in the random sample and clarity regarding the support of $D$ conditional on $X = x$. We omit writing these as formal conditions for the sake of brevity.

Goodman-Bacon, Andrew (2021). “Difference-in-differences with variation in treatment timing”. *Journal of Econometrics* 225.2, pp. 254–277.