Using Poisson Regression for Difference-in-Differences

Here we go, in our attempt to accomodate low-frequency count data models in a DiD setup.

First, notice that the standard parallel trends assumption is not really appealing in this context because it does not "respect" the non-negativity nature of the conditional expectations. To see this, note that the "standard" parallel trends assumption states that

$$\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]-\mathbb{E}\left[Y_{t=1}(\infty)|G=2\right]=\mathbb{E}\left[Y_{t=2}(\infty)|G=\infty\right]-\mathbb{E}\left[Y_{t=1}(\infty)|G=\infty\right].$$

Isolating $\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]$, we get that

$$\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]=\mathbb{E}\left[Y_{t=1}(\infty)|G=2\right]+\mathbb{E}\left[Y_{t=2}(\infty)|G=\infty\right]-\mathbb{E}\left[Y_{t=1}(\infty)|G=\infty\right].$$

Since $Y_{t=2}(\infty)$ is non-negative, the left hand side of the previous equation must also be non-negative. However, this is not impose by the parallel trends assumption: $\mathbb{E}\left[Y_{t=1}(\infty)|G=2\right]+\mathbb{E}\left[Y_{t=2}(\infty)|G=\infty\right]-\mathbb{E}\left[Y_{t=1}(\infty)|G=\infty\right]$ can be negative.

How can we bypass this issue?

To bypass this potential limitation, we need to look for an alternative assumption (with potential different empirical content). In this "non-negative world", we can look for a "ratios-in-ratios" type of model where we assume "parallel relative trends":

$$\frac{\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]}{\mathbb{E}\left[Y_{t=1}(\infty)|G=2\right]}=\frac{\mathbb{E}\left[Y_{t=2}(\infty)|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}(\infty)|G=\infty\right]}.$$

Again, isolating $\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]$, we get that

$$\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]=\mathbb{E}\left[Y_{t=1}(\infty)|G=2\right]\cdot\frac{\mathbb{E}\left[Y_{t=2}(\infty)|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}(\infty)|G=\infty\right]},\tag{1}$$

which is now guaranteed to be non-negative (as long as $Y_{t=1}(\infty)$ is greater than zero for *at least* one unit in the untreated group).

Using (1) and a no-anticipation assumption, we have that the ATT *in levels* at time period $t=2$,

$$ATT_{levels}\equiv\mathbb{E}\left[Y_{t=2}(2)|G=2\right]-\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right],$$

is identified by

$$ATT_{levels} = \mathbb{E}\left[Y_{t=2}|G=2\right] - \mathbb{E}\left[Y_{t=1}|G=2\right] \cdot \frac{\mathbb{E}\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}|G=\infty\right]}.$$

We can estimate it by using the analogy principle, i.e, replacingt population expectations by their sample analogues:

$$\widehat{ATT}_{levels} = \mathbb{E}_n\left[Y_{t=2}|G=2\right] - \mathbb{E}_n\left[Y_{t=1}|G=2\right] \cdot \frac{\mathbb{E}_n\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}_n\left[Y_{t=1}|G=\infty\right]}.$$

Alternatively, we may want to analyze the ATT *in relative terms* at time period $t=2$,

$$ATT_{rel} \equiv \frac{\mathbb{E}\left[Y_{t=2}(2)|G=2\right]}{\mathbb{E}\left[Y_{t=2}(\infty)|G=2\right]}.$$

Note that ATT_{rel} would provide a semi-elasticity. Again, using (1) and a no-anticipation assumption, we have that ATT_{rel} is identified by the "ratio-in-ratioss" estimand:

$$\begin{aligned} ATT_{rel} &= \frac{\mathbb{E}\left[Y_{t=2}|G=2\right]}{\mathbb{E}\left[Y_{t=1}|G=2\right] \cdot \frac{\mathbb{E}\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}|G=\infty\right]}} \\ &= \frac{\left(\frac{\mathbb{E}\left[Y_{t=2}|G=2\right]}{\mathbb{E}\left[Y_{t=1}|G=2\right]}\right)}{\left(\frac{\mathbb{E}\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}|G=\infty\right]}\right)}. \end{aligned}$$

Again, by using the analogy principle, we can estimate ATT_{rel} by

$$\widehat{ATT}_{rel} = \frac{\left(\frac{\mathbb{E}_n\left[Y_{t=2}|G=2\right]}{\mathbb{E}_n\left[Y_{t=1}|G=2\right]}\right)}{\left(\frac{\mathbb{E}_n\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}_n\left[Y_{t=1}|G=\infty\right]}\right)}.$$

How can we implement this via Poisson regressions?

I will now focus on whether $\widehat{ATT}_{rel}$ can be estimated via the following Poisson regression specification

$$
\begin{aligned}
Y_{i,t} &\sim Poisson(\mu_{i,t}), \text{ where} \\
\mu_{i,t} &= \exp\left(\alpha + \gamma_{post}1\{t=2\} + \gamma_{group}1\{G_i=2\} + \beta 1\{t=2\}1\{G_i=2\}\right).
\end{aligned}
$$

The question is whether $\widehat{\beta}^{poisson}$ is equal to (a monotonic transformation of) $\widehat{ATT}_{rel}$.

Let's consider a population Poisson regression and play with the expectations to see what happens. We have four parameters $(\alpha, \gamma_{post}, \gamma_{group}, \beta)$, so we will play with our "old friend", group and time-specific expectations:

$$
\begin{aligned}
\mathbb{E}[Y_{t=1}|G=\infty] &= \exp\left(\alpha + \gamma_{post}\underbrace{1\{t=2\}}_{=0} + \gamma_{group}\underbrace{1\{G_i=2\}}_{=0} + \beta\underbrace{1\{t=2\}1\{G_i=2\}}_{=0}\right) \\
&= \exp(\alpha);
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_{t=2}|G=\infty] &= \exp\left(\alpha + \gamma_{post}\underbrace{1\{t=2\}}_{=1} + \gamma_{group}\underbrace{1\{G_i=2\}}_{=0} + \beta\underbrace{1\{t=2\}1\{G_i=2\}}_{=0}\right) \\
&= \exp(\alpha + \gamma_{post});
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_{t=1}|G=2] &= \exp\left(\alpha + \gamma_{post}\underbrace{1\{t=2\}}_{=0} + \gamma_{group}\underbrace{1\{G_i=2\}}_{=1} + \beta\underbrace{1\{t=2\}1\{G_i=2\}}_{=0}\right) \\
&= \exp(\alpha + \gamma_{group});
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_{t=2}|G=2] &= \exp\left(\alpha + \gamma_{post}\underbrace{1\{t=2\}}_{=1} + \gamma_{group}\underbrace{1\{G_i=2\}}_{=1} + \beta\underbrace{1\{t=2\}1\{G_i=2\}}_{=1}\right) \\
&= \exp(\alpha + \gamma_{post} + \gamma_{group} + \beta).
\end{aligned}
$$

Playing with these, we have that

$$\begin{aligned}\frac{\mathbb{E}\left[Y_{t=2}|G=2\right]}{\mathbb{E}\left[Y_{t=1}|G=2\right]} &= \frac{\exp\left(\alpha+\gamma_{post}+\gamma_{group}+\beta\right)}{\exp\left(\alpha+\gamma_{group}\right)} \\ &= \exp\left(\alpha+\gamma_{post}+\gamma_{group}+\beta-\left(\alpha+\gamma_{group}\right)\right) \\ &= \exp\left(\gamma_{post}+\beta\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathbb{E}\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}|G=\infty\right]} &= \frac{\exp\left(\alpha+\gamma_{post}\right)}{\exp\left(\alpha\right)} \\ &= \exp\left(\alpha+\gamma_{post}-\alpha\right) \\ &= \exp\left(\gamma_{post}\right).\end{aligned}$$

Thus,

$$ATT_{rel} = \frac{\left(\frac{\mathbb{E}\left[Y_{t=2}|G=2\right]}{\mathbb{E}\left[Y_{t=1}|G=2\right]}\right)}{\left(\frac{\mathbb{E}\left[Y_{t=2}|G=\infty\right]}{\mathbb{E}\left[Y_{t=1}|G=\infty\right]}\right)} = \frac{\exp\left(\gamma_{post}+\beta\right)}{\exp\left(\gamma_{post}\right)} = \exp\left(\beta\right).$$

This suggest that, to get ATT_{rel} from a Poisson regression, all we need to do is to "exponentiate" β. For instance, if $\beta = 0.01$,

$$ATT_{rel} = \exp\left(0.01\right) = 1.01005,$$

suggesting that the policy led to a 1% increase in the outcome among units who implemented the policy.

If $\beta = 0.5$,

$$ATT_{rel} = \exp\left(0.5\right) = 1.64872,$$

suggesting that the policy led to a 64.87% increase in the outcome among units who implemented the policy.

How can we allow for "covariate-specific" relative trends?

I will let you think this through, using the steps we followed in class. That is the benefits of focusing on the foundations of the solutions. =)

Hope you find this helpful!

Thanks,
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