Selection and parallel trends*

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Abstract

We study the role of selection into treatment in difference-in-differences (DiD) designs. We derive necessary and sufficient conditions for parallel trends assumptions under general classes of selection mechanisms. These conditions characterize the empirical content of parallel trends. For settings where the necessary conditions are questionable, we propose tools for selection-based sensitivity analysis. We also provide templates for justifying DiD in applications with and without covariates. A reanalysis of the causal effect of NSW training programs demonstrates the usefulness of our selection-based approach to sensitivity analysis.

Keywords: causal inference, conditional parallel trends, covariates, difference-in-differences, selection mechanism, sensitivity analysis, time-invariant and time-varying unobservables, treatment effects

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...while the new papers [in the DiD literature] clarify very well the statistical assumptions needed for estimation, effective use of these methods also requires being able to understand what the threats to these assumptions are in different contexts, and to make a plausible rhetorical argument as to why we should think the assumptions hold.

— David McKenzie, *World Bank Development Impact Blog* (McKenzie, 2022)

1 Introduction

Difference-in-differences (DiD) is a widely-used causal inference method. One of the perceived advantages of DiD is that it does not require explicit assumptions on how units select into treatment but instead relies on parallel trends assumptions. However, when justifying DiD in empirical applications, researchers often argue that the treatment is "quasi-randomly" assigned. Although these discussions allude to selection mechanisms, they are often not explicit about what constitutes "quasi-random" assignment, arguably due to the lack of formal guidance.

In this paper, we study parallel trends assumptions through the lens of selection. We have three goals: (i) characterize the empirical content of parallel trends; (ii) propose new approaches to sensitivity analysis that leverage contextual knowledge about selection; (iii) provide templates for justifying parallel trends in practice with and without covariates. Since DiD is applied in a myriad of empirical contexts, we consider general classes of selection mechanisms that accommodate selection on time-invariant unobservables ("fixed effects"), selection on untreated potential outcomes, selection on treatment effects (Roy-style selection), and other economic models of selection.

We first derive necessary and sufficient conditions for parallel trends. These conditions are helpful for understanding the threats to the identification assumptions underlying DiD, which in turn is essential for an "effective use of these methods," as emphasized by McKenzie (2022)'s quote. We first consider a scenario where researchers are not willing to restrict the selection mechanism.[1] We show that absent any restrictions on selection, parallel trends holds if and only if the untreated potential outcome is constant across time up to deterministic mean shifts. This condition is restrictive in many applications: it essentially rules out time-varying unobservables.

This negative result motivates restricting the selection mechanism. We derive necessary conditions for parallel trends under restrictions that can be motivated based on classical examples of selection as well as the information sets available to units at the time of the decision. First, if the units only select on information from the pre-treatment period (*imperfect*

[1] In Appendix C.1, we provide necessary and sufficient conditions under an alternative scenario where researchers are not willing to impose any restrictions on the distribution of unobservables.

foresight), parallel trends implies that the untreated potential outcome satisfies a martingale property. Second, if the units select into treatment based on *fixed effects*, so that selection does not depend on time-varying unobservables, parallel trends implies a stationarity restriction on the mean of the untreated potential outcome conditional on the fixed effects. Under additional assumptions, these two necessary conditions are also sufficient for parallel trends. Taken together, our necessary and sufficient conditions imply that researchers relying on parallel trends assumptions face a trade-off between restrictions on selection into treatment and restrictions on the time-series properties of the outcomes.

Our necessary conditions for parallel trends motivate a selection-based approach to sensitivity analysis. Suppose, for example, that the units have imperfect foresight such that selection depends on pre-treatment unobservables. In this case, as we show, martingale assumptions on the untreated potential outcomes are necessary for parallel trends. Such assumptions may be restrictive in applications, and if they are violated, DiD is biased for the average treatment effect on the treated (ATT). We characterize the ATT under violations of these martingale assumptions in settings with *and* without additional pre-treatment periods. This characterization allows researchers to leverage contextual knowledge about selection to perform sensitivity analyses, to compute bounds on the ATT, and to construct robust confidence intervals.

We also offer a menu of primitive sufficient conditions for justifying parallel trends in empirical applications, building on our necessary conditions. These conditions constitute theory-based templates for making "plausible rhetorical arguments as to why we should think the [parallel trends] assumptions hold" (McKenzie, 2022). More specifically, these conditions can be used to justify parallel trends based on contextual knowledge about selection, such as what units select on and what information sets are available to them at the time of the selection decision.[2] Our primitive sufficient conditions explicitly allow for selection on time-invariant and time-varying unobservables, thus formalizing what one might mean by "quasi-random" assignment in the context of DiD analyses.

Our necessary and sufficient conditions generalize directly to settings with covariates.[3] They demonstrate that parallel trends assumptions conditional on the trajectory of covariates imply combinations of time homogeneity and separability restrictions on how the covariates enter the outcome model, even when selection only depends on time-invariant unobservables. We therefore consider a weaker conditional parallel trends assumption, designed specifically to accommodate nonseparability between observables and unobservables in the outcome

[2] For example, Arellano et al. (2022) document heterogeneity in the information available to individuals regarding their future incomes.

[3] We assume that covariates are not affected by the treatment. See Caetano et al. (2022) for some recent results relaxing this assumption.

model. We provide a menu of sufficient conditions for this weaker conditional parallel trends assumption and establish connections between these selection-based conditions and identification assumptions in the literature on nonseparable panel data models.

We illustrate the usefulness of the selection-based approach to sensitivity analysis by reanalyzing the causal effect of the NSW training programs using DiD methods. Selection on unobservables in the pre-treatment period (*imperfect foresight*) is a major concern when evaluating training programs. Our sensitivity analysis allows us to assess the sensitivity of DiD with respect to violations of the martingale assumption necessary for parallel trends under imperfect foresight. We find that the DiD estimates without covariates not only differ substantially from the experimental benchmark, but are also very sensitive to violations of the martingale property and thus parallel trends. Incorporating covariates into the analysis reduces the estimated bias relative to the experimental benchmark and also renders the results more robust.

Related literature. This paper contributes to several branches of the literature on causal inference using panel data. Our first contribution is to the classical literature on canonical DiD setups. See, e.g., Ashenfelter (1978), Ashenfelter and Card (1985), Heckman and Robb (1985), Card (1990), Card and Krueger (1994), Meyer et al. (1995), and Angrist and Krueger (1999) for early developments, and Section 2 of Lechner (2010) for a historical perspective. Our contribution is to provide foundations for the parallel trends assumption to hold in non-experimental settings, where selection into treatment may depend on time-invariant and time-varying unobservables.

Our second contribution is to the more recent literature on DiD methods. See, e.g., de Chaisemartin and D'Haultfœuille (2023) and Roth et al. (2023) for surveys. Our paper is most closely related to Roth and Sant'Anna (2023), Arkhangelsky et al. (2021), and Arkhangelsky and Imbens (2022), though our focus greatly differs from theirs. Roth and Sant'Anna (2023) discuss necessary and sufficient conditions under which the parallel trends assumption is satisfied for all (monotonic) transformations of the untreated potential outcome. We, on the other hand, take the outcome model (and thus the specific transformation) as given and study the connection between parallel trends and selection into treatment. Arkhangelsky et al. (2021) and Arkhangelsky and Imbens (2022) propose doubly robust estimation methods that leverage restrictions on outcome models and/or selection models with unconfoundedness-type restrictions; see also Athey et al. (2021). Our results complement theirs as we maintain the parallel trends assumption and discuss the types of restrictions on selection compatible with it. Moreover, our analysis shows that parallel trends is compatible with various types of selection on unobservables, unlike standard unconfoundedness assumptions (e.g., Imbens, 2004; Imbens and Wooldridge, 2009).

Our third contribution is to the literature on sensitivity analysis, partial identification, and robust inference under violations of parallel trends. Our approach differs from the methods in Manski and Pepper (2018), Ban and Kédagni (2023), and Rambachan and Roth (2023) in that we explicitly rely on assumptions on selection that can be motivated from contextual knowledge about a given empirical setting. Our sensitivity analysis can be performed with and without additional pre-treatment periods. Relative to the existing literature, we use the additional pre-treatment periods to learn about the time-series properties of the outcomes, instead of directly making assumptions about how the parallel trends violation changes over time. For these reasons, our sensitivity analysis complements these existing approaches. Our selection-based approach also differs from the analysis by Marx et al. (2023). They derive partial identification results under monotone treatment selection assumptions on the untreated potential outcome, which they motivate using an economic model of learning with binary outcomes. By contrast, we directly exploit necessary conditions for parallel trends under restrictions on the selection mechanism.

Our fourth contribution is to the literature imposing explicit selection and/or outcome models to develop and compare different methods for estimating treatment effects, including DiD (e.g., Ashenfelter and Card, 1985; Heckman and Robb, 1985; Card and Hyslop, 2005; Chabé-Ferret, 2015; Blundell and Dias, 2009; de Chaisemartin and D'Haultfœuille, 2018; Verdier, 2020; Marx et al., 2023). These selection mechanisms were developed for economic models, some of which are tailored to applications such as job training and technology adoption. Our results complement this strand of the literature in several ways. First, our necessary and sufficient conditions are derived for general selection and outcome models that nest models considered in this literature. Our conditions thus clarify trade-offs between assumptions on selection and time-varying unobservables that are relevant for those models. Second, our primitive sufficient conditions nest several of the existing application-specific restrictions. Third, we provide results for general nonseparable models and clarify the role of covariates in the context of parallel trends assumptions. It is worth noting that while most papers in this literature examine sharp DiD designs, as we do, de Chaisemartin and D'Haultfœuille (2018) and Marx et al. (2023) also consider fuzzy DiD designs.

Finally, we establish an explicit connection between DiD and the literature on nonseparable panel models.[4] A strand of this literature has analyzed the identification of average

[4]See, e.g., Altonji and Matzkin (2005); Athey and Imbens (2006); Bester and Hansen (2009); Hoderlein and White (2012); Chernozhukov et al. (2013); Arellano and Bonhomme (2016); Ghanem (2017). This work extends notions of fixed effects and correlated random effects that originated in the linear model (Mundlak, 1961, 1978; Chamberlain, 1982, 1984). Recent surveys (Arellano and Honoré, 2001; Arellano and Bonhomme, 2011) and textbook treatments (Arellano, 2003; Wooldridge, 2010) further describe the role of restrictions on

effects either by allowing for fixed effects and imposing time homogeneity (e.g. Hoderlein and White, 2012; Chernozhukov et al., 2013) or restricting individual heterogeneity via nonparametric correlated random effects assumptions (e.g. Altonji and Matzkin, 2005; Bester and Hansen, 2009). We show that our sufficient conditions for parallel trends imply combinations of time homogeneity and (correlated) random effects restrictions. Our results demonstrate how restrictions on the selection mechanism can be used to justify identification assumptions in the nonseparable panel literature.

Notation. For a random vector W_{it}, where $i = 1, \dots, n$ and $t = 1, 2$, we denote its time series by $W_i \equiv (W_{i1}, W_{i2})$.[5] We use F_W to denote the distribution of the random vector W. Let $f(z, w)$ be a function defined on $\mathcal{Z} \times \mathcal{W}$. We say that $f(z, w)$ is a trivial function of w if $f(z, w) = f(z, w') = h(z)$ for all $z \in \mathcal{Z}$, $w \neq w'$, and $(w, w') \in \mathcal{W}^2$. We say that $f(z, w)$ is a symmetric function in z and w if $f(z, w) = f(w, z)$ for all $(z, w) \in \mathcal{Z} \times \mathcal{W}$. For a vector W_i, W_i^j is the j^{th} element of W_i. We use the notation $\overset{d}{=}$ to denote equality of distribution. For random variables, X_i, Z_i, and W_i, $Z_i|W_i, X_i \overset{d}{=} Z_i|X_i, W_i$ denotes that $F_{Z_i|W_i,X_i}(z|w, x) = F_{Z_i|X_i,W_i}(z|w, x)$ for $(z, w, x) \in \mathcal{Z} \times \mathcal{W} \times \mathcal{X}$.

2 Setup, selection mechanism, and examples

We consider the classical DiD setup with two groups and two periods and abstract from covariates. We discuss the role of covariates in Section 6 and generalize our results to DiD designs with multiple groups and multiple periods in Appendix C.2. Let D_{it} and Y_{it} denote the treatment status and outcome for unit $i \in \{1, \dots, n\}$ in period $t \in \{1, 2\}$. Here the index i refers to the unit making the decision to select into treatment. This could be an individual or a more aggregate administrative unit, such as county or state. See Appendix B for a discussion of DiD designs where the data are available at the disaggregate level (e.g., individual level), while the selection decision is made at the aggregate level (e.g., state level). The treatment group ($G_i = 1$) selects the treatment path $D_i = (0, 1)$; the control group ($G_i = 0$) selects $D_i = (0, 0)$. The potential outcomes with and without the treatment are $Y_{it}(1)$ and $Y_{it}(0)$, respectively.[6]

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time and individual heterogeneity in linear and nonlinear models. Such restrictions have been imposed in the context of identification in limited dependent variable models (e.g. Manski, 1987; Honoré, 1993; Kyriazidou, 1997; Honoré and Kyriazidou, 2000a,b) and random coefficient models (e.g. Chamberlain, 1992; Graham and Powell, 2012; Arellano and Bonhomme, 2012). Nonparametric identification of panel models with additivity restrictions has been examined, e.g., in Evdokimov (2010) and Freyberger (2017).

[5] We define all vectors in this paper as row vectors.

[6] To focus attention on the role of the parallel trends assumption, we assume that there are no anticipatory effects. This is a standard assumption in the DiD literature. See, for example, Roth et al. (2023) for a discussion.

We consider the standard parallel trends assumption. Throughout the paper, we assume that all relevant moments exist and $\{Y_{i1}(0), Y_{i2}(0), G_i\}$ is i.i.d. across i.

Assumption PT. *The (unconditional) parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0].$$

Under Assumption PT, the average treatment effect on the treated group in period $t = 2$, $\text{ATT} \equiv E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1]$, is identified from the "difference-in-differences" as follows:

$$\text{ATT} = E[Y_{i2} - Y_{i1}|G_i = 1] - E[Y_{i2} - Y_{i1}|G_i = 0] \equiv \text{DiD}\,.$$

We work with a general nonseparable model for $Y_{it}(0)$,

$$Y_{it}(0) = \xi_t(\alpha_i, \varepsilon_{it}), \quad i = 1, \ldots, n, \quad t = 1, 2, \tag{1}$$

where α_i, ε_{i1}, and ε_{i2} are finite-dimensional vector-valued random variables, and $\xi_t(\cdot)$ is an unrestricted time-varying function. The outcome model (1), while not imposing any restrictions on $Y_{it}(0)$, allows us to distinguish between time-invariant and time-varying unobservables. This is necessary to define selection mechanisms that can directly depend on these unobservables. If, instead, we were to work directly with potential outcomes, this would rule out important examples of selection mechanisms such as selection on time-invariant unobservables (e.g., Ashenfelter and Card, 1985).

We consider a general class of selection mechanisms in which units select into treatment based on $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ as well as a vector of additional time-invariant and time-varying random variables, $(\nu_i, \eta_{i1}, \eta_{i2})$,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}). \tag{2}$$

This selection mechanism accommodates many different types of selection, including random assignment, selection on fixed effects, selection on untreated potential outcomes, selection on treatment effects, and other economic models of selection (e.g. Heckman and Robb, 1985; Chabé-Ferret, 2015; Marx et al., 2023). Note that since $G_i = D_{i2}$, $g(\cdot)$ can be equivalently viewed as the selection mechanism for D_{i2}. Let $\mathcal{G}_{\text{all}}$ denote the set of all selection mechanisms $g(\cdot)$ mapping from the support of the unobservables to $\{0, 1\}$.

Throughout the paper, we will come back to the following three leading examples of selection, specifically selection on outcomes, on treatment effects, and on fixed effects.

Example 2.1 (Selection on outcomes)**.** *We consider a class of threshold-crossing selection mechanisms, generalizing the selection mechanisms analyzed in Ashenfelter and Card*

(1985), who study the effect of training programs on earnings. Let ω_i denote the information set available to the units when deciding whether to participate in the training program and consider the following mechanism,

$$G_i = 1\left\{E[Y_{i1}(0) + \beta Y_{i2}(0)|\omega_i] \leq E[C_{i2}|\omega_i]\right\}, \tag{3}$$

where $\beta \in [0,1]$ is a discount factor, G_i indicates participation in a job training program, $Y_{it}(0)$ denotes untreated potential earnings, C_{i2} is the individual-specific threshold, which is assumed to be an element of η_{i2}. The selection mechanism (3) can be expressed as $G_i = \check{g}(\omega_i)$ and is therefore a special case of the mechanism (2) if ω_i is a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})$. □

Example 2.2 (Selection on treatment effects (Roy-style selection))**.** *Suppose that units select into the treatment if the expected gains from treatment given the information set ω_i, $E[Y_{i2}(1) - Y_{i2}(0)|\omega_i]$, exceed the expected cost of treatment, $E[C_{i2}|\omega_i]$,*

$$G_i = 1\{E[Y_{i2}(1) - Y_{i2}(0)|\omega_i] \geq E[C_{i2}|\omega_i]\}. \tag{4}$$

The selection mechanism (4) is again a special case of mechanism (2) if ω_i is a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})$. This example shows that it is important to allow $g(\cdot)$ to depend on a vector of additional unobservables, such that we can allow η_{i2} (and thus the information set) to include $(Y_{i2}(1), C_{i2})$. □

Example 2.3 (Selection on fixed effects)**.** *DiD methods have traditionally been motivated using two-way fixed effects models. Fixed effects assumptions allow for unrestricted dependence between time-invariant unobservables and the regressors, thereby implicitly allowing for selection on time-invariant unobservables.*[7] *The general selection mechanism (2) accommodates this classical type of selection if $g(\cdot)$ is a trivial function of $(\varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2})$. A simple example is $G_i = 1\{\alpha_i \leq c\}$, which corresponds to the selection mechanism on p.650 in Ashenfelter and Card (1985).* □

Remark 2.1 (Parallel trends and functional form)**.** *Throughout this paper, we take the functional form of the outcome as given. We thereby abstract from the issues arising from the sensitivity of DiD to functional form specification; see Roth and Sant'Anna (2023) for a discussion.* □

[7] See, e.g., Chamberlain (1984); Arellano (2003); Evdokimov (2010); Wooldridge (2010); Hoderlein and White (2012); Chernozhukov et al. (2013).

3 Necessary and sufficient conditions for parallel trends

3.1 No restrictions on selection

To better understand the implications of parallel trends, we derive necessary and sufficient conditions for this assumption. We start by analyzing a scenario where researchers are not willing to make any assumptions on the selection mechanism so that parallel trends needs to hold for all selection mechanisms.

To ensure non-degeneracy of the selection mechanisms we use to derive necessary and sufficient conditions for parallel trends, we impose the following weak regularity condition.

Assumption SEL. *There exists a component of ν_i, labeled ν_i^1 (w.l.o.g.), such that $\nu_i^1 \perp\!\!\!\perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ and $P(\nu_i^1 > c) \in (0,1)$ for some $c \in \mathbb{R}$.*

Assumption SEL requires that one of the unobservables in the selection mechanism is independent of the unobservable determinants of $Y_{it}(0)$ for $t = 1, 2$. Intuitively, this condition just requires that there is some random shock affecting a unit's decision to select into treatment.

The following proposition presents a necessary and sufficient condition for parallel trends holding for all selection mechanisms. To simplify exposition, we use $\dot{Y}_{it}(0)$ to denote the centered potential outcome without the treatment, $\dot{Y}_{it}(0) \equiv Y_{it}(0) - E[Y_{it}(0)]$, for $t = 1, 2$.

Proposition 3.1 (Necessary and sufficient condition for $g \in \mathcal{G}_{\text{all}}$)**.** *Suppose that Assumption SEL holds and either $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$ or $P(\dot{Y}_{i2}(0) < \dot{Y}_{i1}(0)) < 1$. Then, Assumption PT holds for all $g \in \mathcal{G}_{all}$ satisfying $P(G_i = 1) \in (0,1)$ if and only if $\dot{Y}_{i1}(0) = \dot{Y}_{i2}(0)$ a.s.*

Together with Assumption SEL, $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$ (or $P(\dot{Y}_{i2}(0) < \dot{Y}_{i1}(0)) < 1$) implies that the selection mechanism we use to prove the "only-if" direction of the proof is non-degenerate. These conditions are not restrictive in applications since they merely rule out that the supports of the demeaned potential outcomes are disjoint.

To interpret the necessary and sufficient condition in Proposition 3.1, it is helpful to rewrite it as

$$Y_{i2}(0) = Y_{i1}(0) + E[Y_{i2}(0) - Y_{i1}(0)].$$

This shows that absent any restrictions on selection, parallel trends implies that the potential outcomes are constant over time, except for common mean shifts. Given that this condition is implausible in many applications, we consider restricted classes of selection mechanisms in Section 3.2.

3.2 Restricted selection mechanisms

Motivated by Proposition 3.1, we consider two restricted classes of selection mechanisms. These classes of mechanisms are directly related to and motivated by the information sets available to the units when making the decision to select into the treatment.

First, we examine a class of selection mechanisms in which individuals have *imperfect foresight* so that selection depends on the time-invariant and pre-treatment unobservables,

$$\mathcal{G}_{\text{if}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_2, t_2)\}.$$

In Example 2.1, $\mathcal{G}_{\text{if}}$ captures settings where individuals know their permanent income component, α_i, and the pre-treatment idiosyncratic earnings shock, ε_{i1}, but not the unobservables from the post-treatment period, specifically ε_{i2} and C_{i2}. For empirical evidence on the heterogeneity in income uncertainty faced by different individuals, see, e.g., Arellano et al. (2022). In Example 2.2, assuming that $g \in \mathcal{G}_{\text{if}}$ requires that individuals do not know their treatment effects, $Y_{i2}(1) - Y_{i2}(0)$, and costs, C_{i2}, while their information set can contain all time-invariant and pre-treatment unobservables $(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})$.

Second, we consider a class of mechanisms where selection only depends on the *fixed effects* (α_i, ν_i),

$$\mathcal{G}_{\text{fe}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_1, e_2, t_1, t_2)\}.$$

The class of selection mechanisms $\mathcal{G}_{\text{fe}}$ captures the classical scenario of selection on fixed effects. Assuming that $g \in \mathcal{G}_{\text{fe}}$ is plausible if either the units' information set only contains the time-invariant unobservables in Examples 2.1 and 2.2, so that $\omega_i = (\alpha_i, \nu_i)$, or if selection is directly based on fixed effects as in Example 2.3.

The next two propositions provide necessary conditions for parallel trends when the selection mechanism belongs to $\mathcal{G}_{\text{if}}$ and $\mathcal{G}_{\text{fe}}$, respectively.

Proposition 3.2 (Necessary condition for $g \in \mathcal{G}_{\text{if}}$)**.** *Suppose that Assumption SEL holds and either $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$ or $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] < \dot{Y}_{i1}(0)) < 1$. If Assumption PT holds for all $g \in \mathcal{G}_{if}$ satisfying $P(G_i = 1) \in (0, 1)$, then $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ a.s.*

Proposition 3.3 (Necessary condition for $g \in \mathcal{G}_{\text{fe}}$)**.** *Suppose that Assumption SEL holds and either $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$ or $P(E[\dot{Y}_{i2}(0)|\alpha_i] < E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$. If Assumption PT holds for all $g \in \mathcal{G}_{fe}$ satisfying $P(G_i = 1) \in (0, 1)$, then $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ a.s.*

The two propositions demonstrate that while parallel trends is compatible with the presence of time-varying unobservables under the restricted classes of selection mechanisms, it implies time-series restrictions on $\dot{Y}_{it}(0)$. It is helpful to interpret the necessary conditions under a simple linear two-way model for $Y_{it}(0)$,

$$Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}, \quad E[\varepsilon_{it}] = 0. \tag{5}$$

Under this model, the necessary condition in Proposition 3.2 becomes $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$, a martingale-type property that implies $\varepsilon_{i2} = \varepsilon_{i1} + \zeta_{i2}$, where ζ_{i2} is an innovation satisfying $E[\zeta_{i2}|\alpha_i, \varepsilon_{i1}] = 0$.[8] The necessary condition in Proposition 3.3 simplifies to $E[\varepsilon_{i1}|\alpha_i] = E[\varepsilon_{i2}|\alpha_i]$, a time homogeneity assumption on the conditional mean. In general, the stability of the conditional mean is implied by (and weaker than) the textbook strict exogeneity assumption, $E[\varepsilon_{it}|G_i, \alpha_i] = 0$, since in our framework selection may depend on additional unobservables $(\nu_i, \eta_{i1}, \eta_{i2})$.

The necessary conditions in Propositions 3.2 and 3.3 do not imply parallel trends in general due to the presence of the additional unobservables $(\nu_i, \eta_{i1}, \eta_{i2})$. The following proposition provides simple sufficient conditions in terms of the conditional distribution of the additional unobservables entering the selection mechanism under which these necessary conditions are also sufficient.

Proposition 3.4 (Sufficient conditions for $\mathcal{G}_{\text{if}}$ and $\mathcal{G}_{\text{fe}}$)**.** *Suppose that $P(G_i = 1) \in (0, 1)$.*

(i) Suppose that $g \in \mathcal{G}_{if}$. If $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$, then $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ implies Assumption PT.

(ii) Suppose that $g \in \mathcal{G}_{fe}$. If $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} \nu_i|\alpha_i$, then $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ implies Assumption PT.

Taken together, our necessary conditions demonstrate trade-offs between restrictions on selection into treatment and restrictions on the time-series properties of potential outcomes. In particular, these results highlight the role of restrictions on time-varying unobservables, either in terms of how they vary over time or how they determine selection. As a result, researchers using DiD approaches cannot avoid making meaningful and nontrivial assumptions on selection and time-varying unobservables.

3.3 Necessary and sufficient conditions: extensions

Here, we briefly summarize two extensions. See Appendix C for details.

[8] The result in Proposition 3.2 relates to the consistency of the first-differences estimator under violations of strict exogeneity when the idiosyncratic shocks follow a unit root. In fact, under sequential exogeneity, selection into treatment depends on the lagged outcome and the time-invariant unobservable such that $G_i = g(\alpha_i, \varepsilon_{i1})$ (Chamberlain, 2022) and, thus, $E[\varepsilon_{i2}|G_i, \alpha_i, \varepsilon_{i1}] = E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}]$. If, in addition, $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$, then it follows that $E[\varepsilon_{i2} - \varepsilon_{i1}|G_i, \alpha_i, \varepsilon_{i1}] = 0$, which implies that $E[\varepsilon_{i2} - \varepsilon_{i1}|G_i] = E[\varepsilon_{i2} - \varepsilon_{i1}]$ and thus Assumption PT in the separable model (5). We thank Stéphane Bonhomme for pointing out this connection.

3.3.1 Parallel trends for any distribution

In Appendix C.1, we provide necessary and sufficient conditions for an alternative scenario where researchers are not willing to restrict the distribution of unobservables. Specifically, suppose researchers want parallel trends to hold for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$, where $\mathcal{F}$ is a complete class of distributions.[9] We show that Assumption PT holds for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$ if and only if

$$P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1) \text{ a.s. for all } F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}.$$

That is, parallel trends (holding for all distributions of unobservables) is equivalent to selection being independent of all the unobservable determinants of the untreated potential outcome.

3.3.2 Multiple periods and multiple groups

In Appendix C.2, we extend our results to DiD designs with multiple periods and multiple groups.[10] Specifically, we consider a staggered adoption setting with T periods, where no units are treated at $t = 1$ and some units remain untreated at $t = T$. The group indicator G_i denotes the first period in which units select into the treatment. We set $G_i = \infty$ for the never-treated units so that $G_i \in \{2, \ldots, T, \infty\}$.

We provide three necessary conditions for the standard parallel trends assumption on the never-treated potential outcome $Y_{it}(\infty)$,

$$E[Y_{it}(\infty) - Y_{i(t-1)}(\infty) \,|\, G_i = g] = E[Y_{it}(\infty) - Y_{i(t-1)}(\infty)|G_i = \infty] \quad \text{for all } (g, t). \tag{6}$$

These conditions can be viewed as natural generalizations of Propositions 3.1, 3.2, and 3.3 to the multiple-group, multiple-period case.

4 Sensitivity analysis under assumptions on selection

The necessary conditions in Section 3 demonstrate that if we allow for selection on time-varying shocks, parallel trends implies strong restrictions on the time-series properties of the outcomes. Here we build on these results by developing tools for sensitivity analysis

[9] Intuitively, completeness of $\mathcal{F}$, which is formally defined in Definition C.1, requires that the class of possible distributions of unobservables is "rich enough." This condition is trivially satisfied if $\mathcal{F}$ is unrestricted.

[10] Our setup and notation build on Callaway and Sant'Anna (2021), Sun and Abraham (2021), and Roth et al. (2023).

that allow researchers to exploit contextual information about selection in the presence of deviations from the relevant necessary conditions (and thereby violations of parallel trends).

4.1 Identifying the ATT under deviations from the martingale assumption

To motivate our identification approach, we decompose the DiD estimand as[11]

$$\begin{aligned} \text{DiD} &= \text{ATT} + E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|G_i = 1] - E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|G_i = 0] \\ &\equiv \text{ATT} + \Delta_{\text{post}}. \end{aligned} \tag{7}$$

This decomposition shows that the DiD estimand is equal to the sum of the ATT and the bias term Δ_{post}, which captures the bias due to violations of Assumption PT. Below, we characterize the bias Δ_{post} and thus the ATT under assumptions on selection.

In many applications, individuals select into treatment based on time-invariant and time-varying unobservables in the pre-treatment periods (*imperfect foresight*). For example, individuals may select into the training if their pre-treatment earnings fall below a certain cutoff (Ashenfelter and Card, 1985), as in Example 2.1 with $\beta = 0$ and $\omega_i = (\alpha_i, \varepsilon_{i1})$. Alternatively, they might select into job training programs if their expected net gain from the training conditional on these (pre-treatment) unobservables is greater than zero, as in Example 2.2 with $\omega_i = (\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})$. The necessary conditions in Section 3 show that parallel trends implies strong restrictions on the time-series properties of $\dot{Y}_{it}(0)$ under imperfect foresight. Here we characterize the ATT when these time-series restrictions are violated.

We consider a setting with one additional pre-treatment period, $t = 0$, in which no units are treated so that $Y_{i0} = Y_{i0}(0)$ for $i = 1, \ldots, n$. We allow selection to also depend on the shocks in period $t = 0$, $G_i = g(\alpha_i, \varepsilon_i^1, \varepsilon_{i2}, \nu_i, \eta_i^1, \eta_{i2})$, where $\varepsilon_i^1 \equiv (\varepsilon_{i0}, \varepsilon_{i1})$ and $\eta_i^1 \equiv (\eta_{i0}, \eta_{i1})$, and modify the definition of $\mathcal{G}_{\text{if}}$ accordingly,

$$\mathcal{G}_{\text{if}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e^1, e_2, v, t^1, t_2) \text{ is a trivial function of } (e_2, t_2)\}.$$

We study identification of the ATT under the following imperfect foresight assumption.

Assumption IF. *The following conditions hold: (i) $g \in \mathcal{G}_{if}$ and (ii) $(\nu_i, \eta_i^1)|\alpha_i, \varepsilon_i^1, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_i^1)|\alpha_i, \varepsilon_i^1$.*

Assumption IF embeds two conditions: (i) the selection mechanism does not directly depend on future shocks, and (ii) conditions on the distribution of the additional pre-

[11] Rambachan and Roth (2023, Section 2.2) use the decomposition: DiD = ATT $+E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1] - E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0]$. We write this decomposition in terms of demeaned outcomes here to relate the bias to the results in Section 3.

treatment unobservables in periods $t \in \{0,1\}$. Together, these two conditions imply that $E[G_i|\alpha_i, \varepsilon_i^1, \varepsilon_{i2}] = E[G_i|\alpha_i, \varepsilon_i^1]$. While Assumption IF implies that $Y_{i2}(0) \perp\!\!\!\perp G_i|\alpha_i, \varepsilon_{i1}$, we emphasize that it does not imply unconfoundedness conditional on $Y_{i1}(0)$, $Y_{i2}(0) \perp\!\!\!\perp G_i|Y_{i1}(0)$, except under specialized and restrictive conditions.[12]

Using the same arguments as in Proposition 3.2, one can show that Assumption PT holding for all $g \in \mathcal{G}_{\text{if}}$ implies that $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] = \dot{Y}_{i1}(0)$. This motivates relating the bias Δ_{post} to deviations from the martingale condition. To this end, we consider the following general class of relaxations of this condition. Other choices are possible.

Assumption REL. *The following relaxation of the martingale condition holds:*[13]

$$E[\dot{Y}_{it}(0)|\alpha_i, \varepsilon_{i0}, \ldots, \varepsilon_{i(t-1)}] = \phi(\dot{Y}_{i(t-1)}(0); \rho_t), \quad i = 1, \ldots, n, \quad t = 1, 2,$$

where $\phi(\cdot; \rho_t)$ is a function that is known up to the (time-varying) parameter ρ_t, which may be infinite-dimensional.

Since Assumption REL is a relaxation of the martingale property, it imposes a nonlinear AR(1) model with time-varying coefficients on $\dot{Y}_{it}(0)$. If $\phi(\cdot; \rho_t)$ is the identity function, Assumption REL reduces to the martingale assumption, $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] = \dot{Y}_{i1}(0)$. If $\phi(y; \rho_t)$ is linear so that

$$E[\dot{Y}_{it}(0)|\alpha_i, \varepsilon_{i0}, \ldots, \varepsilon_{i(t-1)}] = \rho_t \dot{Y}_{i(t-1)}(0), \tag{8}$$

the parameter ρ_t is simply the (time-varying) AR(1) coefficient. Since Assumption REL is imposed on the demeaned potential outcomes, it allows for location shifts in $Y_{it}(0)$.

In Assumption REL, ρ_1 can be identified from the pre-treatment data by noting that $E[\dot{Y}_{i1}(0)|\dot{Y}_{i0}(0)] = E[E[\dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i0}]|\dot{Y}_{i0}(0)] = \phi(\dot{Y}_{i0}(0); \rho_1)$. For example, under linearity, ρ_1 is identified as the coefficient of a population regression of $\dot{Y}_{i1}$ on $\dot{Y}_{i0}$. Thus, the key unobservable quantity is ρ_2, which parametrizes the deviation of the martingale property in the post-treatment period. The parameter ρ_2 can be interpreted as a measure of the persistence of the potential outcomes. Such interpretation is particularly straightforward when the deviation is linear, as in (8). Since ρ_1 is identified, it can be used to gauge the value or a range of values for ρ_2. We further discuss this point below and provide an empirical illustration in Section 7.

[12]For instance, this unconfoundedness condition holds under Assumption IF if, in addition, selection into treatment is solely a function of $Y_{i1}(0)$, specifically $G_i = g(Y_{i1}(0))$. Of course, if a researcher had this *a priori* knowledge about the selection mechanism, then they should use a method that exploits the unconfoundedness assumption rather than using DiD.

[13]Assumption REL yields a linear autoregressive model when $\phi(\cdot; \rho_t)$ is linear. This class of models has been studied extensively in the time series literature under restrictions on the heterogeneity of the coefficient (e.g., Nicholls and Quinn, 1982; Regis et al., 2022).

The following proposition characterizes the ATT under Assumption REL.

Proposition 4.1 (ATT under violations of the martingale assumption)**.** *Suppose that Assumption IF holds. Suppose further that $P(G_i = 1) \in (0, 1)$. If Assumption REL holds, then* $\text{ATT} \equiv \text{ATT}(\rho_2) = \text{DiD} - \Delta_{post}(\rho_2)$*, where*

$$\Delta_{post}(\rho_2) = \frac{E[G_i(\phi(\dot{Y}_{i1}; \rho_2) - \dot{Y}_{i1})]}{P(G_i = 1)P(G_i = 0)}.$$

Proposition 4.1 can be used in at least three related but different ways. First, if ρ_2 is known, Proposition 4.1 point-identifies the ATT under violations of the martingale assumption (and thus Assumption PT). This motivates performing sensitivity analyses by plotting the ATT as a function of ρ_2, as we illustrate in the empirical application in Section 7. Such sensitivity analyses can be performed without additional pre-treatment periods. However, when data on additional pre-treatment periods are available, we recommend informing the range of values for ρ_2 in the sensitivity analysis based on ρ_1, the parameter governing the martingale relaxation in the pre-treatment period.

Second, given a range of possible values for ρ_2, $\left[\underline{\rho}_2, \overline{\rho}_2\right]$, the ATT is partially identified, and the identified set is a closed interval:

$$\text{ATT} \in \left\{\text{ATT}(\rho_2) : \rho_2 \in \left[\underline{\rho}_2, \overline{\rho}_2\right]\right\}.$$

When additional pre-treatment periods are available so that ρ_1 is identified, we recommend using ρ_1 to inform the choice of $\underline{\rho}_2$ and $\overline{\rho}_2$. For example, one can obtain $\underline{\rho}_2$ and $\overline{\rho}_2$ based on restrictions on the change in persistence over time, $\%\Delta_\rho \equiv (\rho_2 - \rho_1)/\rho_1$ (provided that $\rho_1 \neq 0$). Alternatively, one could restrict $|\rho_2/\rho_1|$ or $|\rho_2 - \rho_1|$.[14]

Finally, Proposition 4.1 can be used to construct confidence intervals for the ATT that are robust to violations of the martingale property necessary for parallel trends under imperfect foresight. Such confidence intervals could be constructed, for example, using the approach proposed by Conley et al. (2012).

Remark 4.1 (Incorporating covariates)**.** *The sensitivity analysis extends to settings with covariates in a straightforward manner, since the identification result in Proposition 4.1 remains valid conditional on covariates. See Appendix A for more details and Section 7 for an empirical illustration.* □

[14] We emphasize that this is different from Rambachan and Roth (2023) who restrict the evolution of the parallel trends violation itself. By contrast, we restrict the evolution of a different parameter: the autoregressive parameter ρ_t that governs the persistence of $Y_{it}(0)$.

Remark 4.2 (Multiple periods and groups). *The sensitivity analysis proposed here can be extended to the case with multiple (post-treatment) periods and groups. We outline this extension in Section D.* □

Remark 4.3 (Modeling ρ_t with multiple pre-treatment periods.). *In settings with multiple pre-treatment periods, the identification strategy in this section can be refined. Specifically, one can impose a parametric model for ρ_t and use this model to impute or determine a range for ρ_2. A simple example would be a linear model, $\rho_t = \rho_0 + \rho_1 t$. The more pre-treatment periods are available, the more flexible the model for ρ_t can be.* □

4.2 Sensitivity analysis under linear deviations

When the relaxation of the martingale condition is linear as in equation (8), the characterization of the ATT in Proposition 4.1 simplifies substantially. In this case, the bias of DiD, $\Delta_{\text{post}}(\rho_2)$, is equal to the product of the deviation from the martingale property and the selection bias in the pre-treatment period,

$$\begin{aligned} \text{ATT}(\rho_2) &= \text{DiD} - \Delta_{\text{post}}(\rho_2) \\ &= \text{DiD} - \underbrace{(\rho_2 - 1)}_{\substack{\text{deviation from} \\ \text{martingale property}}} \times \underbrace{(E[Y_{i1}|G_i = 1] - E[Y_{i1}|G_i = 0])}_{\text{selection bias in } t=1}. \end{aligned} \tag{9}$$

Under random assignment, there is no selection bias, and $\Delta_{\text{post}}(\rho_2)$ is equal to zero regardless of the deviation from the martingale property. We can further rewrite the ATT as

$$\text{ATT}(\rho_2) = E[Y_{i2}|G_i = 1] - E[Y_{i2}|G_i = 0] - \rho_2(E[Y_{i1}|G_i = 1] - E[Y_{i1}|G_i = 0]). \tag{10}$$

The expressions for the ATT in equations (9) and (10) provide two alternative interpretations of the identification result in Proposition 4.1. First, we can interpret the result as a bias-correction approach based on an explicit formula for the bias of DiD due to the violation of the martingale property. Second, we can interpret the identification result as a generalized version of DiD in which the pre-treatment difference is multiplied by ρ_2 (as opposed to 1 in classical DiD).

5 Templates for justifying parallel trends in applications

The results in the previous sections illustrate that restrictions on time-varying unobservables are necessary for parallel trends to hold. Here we discuss three sets of sufficient conditions that practitioners can use to justify parallel trends in empirical applications, depending on the assumptions they are willing to impose on the selection mechanism. The exact form of these sufficient conditions depends on the model for the potential outcome in the absence of the treatment. Here, we present the conditions for the separable two-way model (5). See Section 6.2 for sufficient conditions for general nonseparable models.

The first sufficient condition demonstrates a case where selection can depend on both ε_{i1} and ε_{i2}, *and* the untreated potential outcomes can vary across time beyond location shifts. Define the class of symmetric selection mechanisms as

$$\mathcal{G}_{\text{sym}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a symmetric function in } e_1 \text{ and } e_2\}.$$

Assumption SC1. *The following conditions hold: (i)* $g \in \mathcal{G}_{sym}$*, (ii)* $\varepsilon_{i1}, \varepsilon_{i2}|\alpha_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1}|\alpha_i$*, and (iii)* $(\nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i2}, \varepsilon_{i1}$.

In addition to symmetry of the selection mechanism, Assumption SC1 imposes two different types of exchangeability restrictions. First, it requires that the conditional distribution of $(\nu_i, \eta_{i1}, \eta_{i2})$ is exchangeable in ε_{i1} and ε_{i2} after conditioning on α_i. This notion of exchangeability has been employed, for example, in Altonji and Matzkin (2005). Second, it requires the distribution of $(\varepsilon_{i1}, \varepsilon_{i2})$ to be exchangeable conditional on α_i.

The next two sufficient conditions directly build on Propositions 3.2, 3.3, and 3.4.

Assumption SC2. *The following conditions hold: (i)* $g \in \mathcal{G}_{if}$*, (ii)* $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$*, and (iii)* $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$.

Assumption SC3. *The following conditions hold: (i)* $g \in \mathcal{G}_{fe}$*, (ii)* $E[\varepsilon_{i1}|\alpha_i] = E[\varepsilon_{i2}|\alpha_i]$*, and (iii)* $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} \nu_i|\alpha_i$.

The following proposition formally establishes the sufficiency of Assumptions SC1, SC2, and SC3.

Proposition 5.1 (Templates for justifying Assumption PT)**.** *Suppose that* $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$*, where* $E[\varepsilon_{it}] = 0$*, and* $P(G_i = 1) \in (0, 1)$*. Then (i) Assumption SC1 implies Assumption PT, (ii) Assumption SC2 implies Assumption PT, and (iii) Assumption SC3 implies Assumption PT.*

The sufficient conditions SC1, SC2, and SC3 provide practitioners with explicit theory-based templates for justifying parallel trends assumptions. These templates allow researchers to provide, in the words of McKenzie (2022), "plausible rhetorical arguments" based on contextual knowledge about selection. These conditions can be used, for example, in conjunction with the selection mechanisms in Examples 2.1, 2.2, and 2.3. In Section 8, we discuss their practical implications.

6 Covariates and the role of separability

In many applications, parallel trends may only be plausible conditional on covariates (e.g., Heckman et al., 1997; Abadie, 2005; Sant'Anna and Zhao, 2020a; Callaway and Sant'Anna, 2021). Therefore, we study the role of covariates through the lens of selection into treatment. While many existing approaches focus on time-invariant covariates, we explicitly allow for a vector of both time-invariant and time-varying covariates, X_{it}, assuming that X_{it} is not affected by the treatment.[15]

We start by demonstrating that conditional parallel trends assumptions imply separability restrictions with respect to how the covariates can enter the outcome equation. We then provide a set of sufficient conditions for a weaker version of the parallel trends assumption that accommodates nonseparable models and discuss connections to the literature on nonseparable panel data models.

6.1 Conditional parallel trends assumptions imply separability

Suppose that parallel trends holds conditional on the time series of covariates.

Assumption PT-X. *The conditional parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, X_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, X_i] \ a.s.$$

Under Assumption PT-X, the unconditional ATT is identified as

$$E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1] = E\left[\text{ATT}(X_i)|G_i = 1\right] = E\left[\text{DiD}(X_i)|G_i = 1\right],$$

where $\text{ATT}(X_i) \equiv E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_i]$ and $\text{DiD}(X_i) \equiv E[Y_{i2} - Y_{i1}|G_i = 1, X_i] - E[Y_{i2} - Y_{i1}|G_i = 0, X_i]$.

In the presence of covariates, potential outcomes and selection into treatment may naturally depend on them. We therefore consider the following outcome model and selection mechanism,

$$\begin{aligned} Y_{it}(0) &= \xi_t(X_{it}, \alpha_i, \varepsilon_{it}), \\ G_i &= g(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}). \end{aligned}$$

Denote by $\mathcal{G}_{\text{all}}$ the class of all selection mechanisms and define the following restricted classes

[15] See Caetano et al. (2022) for an analysis of settings where covariates can be affected by the treatment.

of selection mechanisms,

$$\begin{aligned}\mathcal{G}_{\text{if}} &= \{g \in \mathcal{G}_{\text{all}} : g(x_1, x_2, a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_2, t_2)\} \\ \mathcal{G}_{\text{fe}} &= \{g \in \mathcal{G}_{\text{all}} : g(x_1, x_2, a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_1, e_2, t_1, t_2)\}.\end{aligned}$$

All the necessary conditions in Section 3 generalize straightforwardly to settings with covariates. Let $\ddot{Y}_{it}(0) \equiv Y_{it}(0) - E[Y_{it}(0)|X_i]$. Assumption PT-X holds for all $g \in \mathcal{G}_{\text{all}}$ if and only if $\ddot{Y}_{i1}(0) = \ddot{Y}_{i2}(0)$. If Assumption PT-X holds for all $g \in \mathcal{G}_{\text{if}}$, then $E[\ddot{Y}_{i2}(0)|X_i, \alpha_i, \varepsilon_{i1}] = \ddot{Y}_{i1}(0)$, and if Assumption PT-X holds for all $g \in \mathcal{G}_{\text{fe}}$, then $E[\ddot{Y}_{i2}(0)|X_i, \alpha_i] = E[\ddot{Y}_{i1}(0)|X_i, \alpha_i]$.

An important practical implication of these necessary conditions is that they imply separability requirements on how the covariates can enter the outcome model, even when selection only depends on time-invariant unobservables and covariates ($g \in \mathcal{G}_{\text{fe}}$). This can be seen by rewriting the corresponding necessary condition as

$$E[Y_{i2}(0)|X_i, \alpha_i] - E[Y_{i1}(0)|X_i, \alpha_i] = E[Y_{i2}(0)|X_i] - E[Y_{i1}(0)|X_i].$$

To illustrate the separability restrictions, consider a generalized random coefficient model (e.g., Chamberlain, 1992) where α_i interacts with X_{it},

$$\xi_t(X_{it}, \alpha_i, \varepsilon_{it}) = \alpha_i \gamma_t(X_{it}) + \lambda_t + \varepsilon_{it}. \tag{11}$$

Here $\gamma_t(\cdot)$ is an arbitrary time-varying function. Even under the assumption that $E[\varepsilon_{it}|X_i, \alpha_i] = 0$, this model generally violates the necessary condition due to the combination of nonseparability between α_i and X_{it} and the time variability in the structural function through $\gamma_t(\cdot)$,

$$E[Y_{i2}(0)|X_i, \alpha_i] - E[Y_{it}(0)|X_i, \alpha_i] = \alpha_i(\gamma_2(X_{i2}) - \gamma_1(X_{i1})) + \lambda_2 - \lambda_1.$$

Allowing for interactions between the unobservable determinants of selection and some covariates is important in applications. Therefore, we consider a weaker conditional parallel trends assumption that allows for such interactions in Section 6.2.

Remark 6.1 (Templates for justifying parallel trends in separable models with covariates)**.** *The discussion in this section shows that Assumption PT-X requires separability between the observable and unobservable determinants of selection in the outcome model. In Appendix E, we provide three sets of primitive sufficient conditions for Assumption PT-X based on the model,* $Y_{it}(0) = \alpha_i + \gamma_t(X_{it}) + \lambda_t + \varepsilon_{it}$*. In this model, the covariates enter in an additively separable manner through the arbitrary and potentially time-varying function* $\gamma_t(\cdot)$*. These sufficient conditions are conditional versions of Assumptions SC1, SC2, and SC3.* □

6.2 A parallel trends assumption for nonseparable models

Motivated by Section 6.1, we consider a weaker (than Assumption PT-X) conditional parallel trends assumption. To define this assumption, we explicitly differentiate between two types of covariates: (i) X_{it}^{μ} are covariates that interact with the unobservable determinants of selection in the outcome model; (ii) X_{it}^{λ} are covariates that do not interact with these unobservables in the outcome model. Both types of covariates can enter the selection mechanism in an arbitrary way. The conditional parallel trends assumption we introduce next holds for subpopulations that experience no change in X_{it}^{μ} and the same trajectory in X_{it}^{λ}.

Assumption PT-NSP. *The (modified) conditional parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \ a.s.$$

Under Assumption PT-NSP, we can no longer identify the ATT, $E[Y_{i2}(1)-Y_{i2}(0)|G_i = 1]$, because we cannot identify the conditional ATT, $E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_i^{\lambda}, X_i^{\mu}]$. Instead, we can identify $E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]$.[16] After integrating out with respect to the distribution of covariates, we can identify the ATT for subpopulations that do not experience changes in X_{it}^{μ},

$$E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_{i1}^{\mu} - X_{i2}^{\mu} = 0].$$

Note that if X_{it}^{μ} is time-invariant, then $X_{i1}^{\mu} = X_{i2}^{\mu}$ holds by definition such that Assumptions PT-X and PT-NSP are equivalent.

In view of Assumption PT-NSP, we consider the following nonseparable model which consists of a time-invariant and time-varying component.

Assumption NSP-X.

$$Y_{it}(0) = \mu(X_{it}^{\mu}, \alpha_i^{\mu}, \varepsilon_{it}^{\mu}) + \lambda_t(X_{it}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{it}^{\lambda}), \quad i = 1, \dots, n, \quad t = 1, 2,$$

where X_{it}^{μ}, X_{it}^{λ}, α_i^{μ}, α_i^{λ}, ε_{it}^{μ}, and $\varepsilon_{it}^{\lambda}$ are finite-dimensional random vectors.

Without further restrictions on the unobservables, the additive structure in Assumption NSP-X is without loss of generality and the superscripts μ and λ are merely labels. Indeed, if $X_{it}^{\mu} = X_{it}^{\lambda}$, $\alpha_i^{\mu} = \alpha_i^{\lambda}$, and $\varepsilon_{it}^{\mu} = \varepsilon_{it}^{\lambda}$, the model is fully nonseparable and time-varying in an arbitrary way. In the following, we use $\mathcal{X}_{\mu}$, $\mathcal{X}_{\lambda}$, $\mathcal{A}$, and $\mathcal{E}$ to denote the supports of X_{it}^{μ}, X_{it}^{λ}, α_i^{μ}, and ε_{it}^{μ}, respectively.

[16] With a slight abuse of notation, we use $(X_{i1}^{\mu} = X_{i2}^{\mu})$ in the conditioning set as a short-hand for $(X_{i1}^{\mu}, X_{i2}^{\mu} = X_{i1}^{\mu})$.

In view of the necessary conditions, it is natural to consider selection based on the unobservables entering $\mu(\cdot)$. We therefore impose the following condition on the projected selection mechanism.

Assumption SEL-CI.

$$E[G_i|X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \alpha_i^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda] = E[G_i|X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu].$$

Assumption SEL-CI allows the projected selection mechanism to depend on all covariates, but only on the unobservables that enter the time-invariant component of the structural function. In view of Assumption SEL-CI, we define

$$\begin{aligned}&\bar{g}(x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, a^\mu, e_1^\mu, e_2^\mu)\\ &\equiv E[G_i|X_{i1}^\mu = x_1^\mu, X_{i2}^\mu = x_2^\mu, X_{i1}^\lambda = x_1^\lambda, X_{i2}^\lambda = x_2^\lambda, \alpha_i^\mu = a^\mu, \varepsilon_{i1}^\mu = e_1^\mu, \varepsilon_{i2}^\mu = e_2^\mu].\end{aligned}$$

We present three sets of sufficient conditions for Assumption PT-NSP. Each set of conditions consists of assumptions on the projected selection mechanism as well as distributional restrictions on the unobservables. Our first sufficient condition allows selection to depend on all covariates as well as the unobservables that enter the time-invariant component of the structural function, while imposing a symmetry restriction on the projected selection mechanism similar to Assumption SC1.

Assumption SC1-NSP. *The following conditions hold:*

(i) $\bar{g}(x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, a^\mu, e_1^\mu, e_2^\mu)$ *is a symmetric function in* e_1^μ *and* e_2^μ*.*

(ii) $(\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\mu, X_i^\lambda, \alpha_i^\mu \overset{d}{=} (\varepsilon_{i2}^\mu, \varepsilon_{i1}^\mu)|X_i^\mu, X_i^\lambda, \alpha_i^\mu$*.*

(iii) $(\alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu) \perp\!\!\!\perp (\alpha_i^\lambda, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda)|X_i^\mu, X_i^\lambda$*.*

Here we require the conditional distribution of $(\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\mu, X_i^\lambda, \alpha_i^\mu$ to be exchangeable. Since the projected selection mechanism depends on $(\alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)$, we require them to be independent of the unobservables entering $\lambda_t(\cdot)$ conditional on (X_i^μ, X_i^λ).

The exchangeability restriction in Assumption SC1-NSP is different from the exchangeability assumption in Altonji and Matzkin (2005). The exchangeability assumption in Altonji and Matzkin (2005) requires the conditional distribution of all unobservables that enter $\mu(\cdot)$ and $\lambda_t(\cdot)$ to be invariant to permutations of covariates in the conditioning set, which is a nonparametric correlated random effects restriction (Ghanem, 2017). By contrast, we assume that the time-varying unobservables are exchangeable conditional on $(X_i^\mu, X_i^\lambda, \alpha_i^\mu)$ without imposing any restrictions on the distribution of $\alpha_i^\mu|G_i, X_i^\mu, X_i^\lambda$.

Next, in the spirit of Assumption SC2, we consider a projected selection mechanism that is a trivial function of ε_{i2}^{μ} in the following sufficient condition.

Assumption SC2-NSP. *The following conditions hold:*

(i) $\bar{g}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}, e_1^{\mu}, e_2^{\mu})$ *is a trivial function of* e_2^{μ}.

(ii) $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}) \perp\!\!\!\perp \Delta_{\mu,i} | X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}$, *where* $\Delta_{\mu,i} \equiv \mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})$.

(iii) $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}) \perp\!\!\!\perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda}) | X_i^{\mu}, X_i^{\lambda}$.

Assumption SC2-NSP.ii implicitly imposes separability conditions on $\mu(\cdot)$ (but not on $\lambda_t(\cdot)$) and restrictions on time series dependence.[17] The independence condition in Assumption SC2-NSP.iii requires that the unobservable determinants of selection are independent of the unobservables that enter $\lambda_t(\cdot)$ conditional on the time series of covariates.

The last sufficient condition restricts the projected selection mechanism to only depend on covariates and the time-invariant unobservables.

Assumption SC3-NSP. *The following conditions hold:*

(i) $\bar{g}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}, e_1^{\mu}, e_2^{\mu})$ *is a trivial function of* e_1^{μ} *and* e_2^{μ}.

(ii) $\varepsilon_{i1}^{\mu} | X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu} \stackrel{d}{=} \varepsilon_{i2}^{\mu} | X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu}$.

(iii) $\alpha_i^{\mu} \perp\!\!\!\perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda}) | X_i^{\mu}, X_i^{\lambda}$.

Assumption SC3-NSP requires the distribution of ε_{it}^{μ}, which enters $\mu(\cdot)$, to be time-invariant conditional on $(\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda})$. The unobservables entering $\lambda_t(\cdot)$, $(\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})$, are required to be independent of the unobservables that determine selection, α_i^{μ}, conditional on $(X_i^{\mu}, X_i^{\lambda})$.

Each of the sufficient conditions consists of three components: (i) a restriction on how/which unobservables determine the projected selection mechanism, (ii) a restriction on the unobservables entering the time-invariant component of the structural function, and (iii) an independence assumption that ensures that the time-varying component of the structural function is independent of G_i conditional on the time series of covariates.

The following proposition formally establishes sufficiency of each set of conditions.

Proposition 6.1 (Sufficient conditions)**.** *Suppose that Assumptions NSP-X and SEL-CI hold and* $P(G_i = 1 | X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}) \in (0,1)$ *a.s. Then (i) Assumption SC1-NSP implies Assumption PT-NSP, (ii) Assumption SC2-NSP implies Assumption PT-NSP, and (iii) Assumption SC3-NSP implies Assumption PT-NSP.*

[17] To see this, note that since $\Delta_{\mu,i} = \mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})$, for $\Delta_{\mu,i}$ to be conditionally independent of $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})$, a sufficient condition would be that $\Delta_{\mu,i}$ is separable in α_i^{μ} and ε_{it}^{μ} as well as the independence of the component that includes ε_{i1}^{μ} and ε_{i2}^{μ} of $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu})$.

Remark 6.2 (Connection to unconfoundedness). *All sufficient conditions in Proposition 6.1 allow for selection on unobservable determinants of the untreated potential outcome. This is in contrast to the unconfoundedness assumptions commonly used in cross-sectional studies (e.g., Imbens, 2004; Imbens and Wooldridge, 2009). Therefore, these results elucidate the differences between conditional parallel trends and unconfoundedness-type assumptions.* □

6.3 Connections to identification assumptions in panel models

DiD methods have traditionally been motivated using two-way fixed effects models. As discussed in Example 2.3, fixed effects assumptions allow for selection on time-invariant unobservables. In this paper, we explicitly analyze the connection between selection mechanisms and the parallel trends assumptions underlying DiD. Therefore, a natural question is how our sufficient conditions relate to the identification assumptions in the nonseparable panel literature.

The literature on nonseparable panel models has considered two broad categories of identification assumptions. First, time homogeneity conditions (e.g., Hoderlein and White, 2012; Chernozhukov et al., 2013) require the distribution of time-varying unobservables to be stationary across time while allowing for unrestricted individual heterogeneity (fixed effects). Second, nonparametric correlated random effects restrictions (e.g., Altonji and Matzkin, 2005; Bester and Hansen, 2009) allow for unrestricted time heterogeneity by imposing restrictions on individual heterogeneity, generalizing the classical notion of correlated random effects (e.g., Mundlak, 1978; Chamberlain, 1984). However, neither category of assumptions is explicit about the selection mechanism and, in particular, about how unobservables determine selection.

The existing identification results based on time homogeneity or correlated random effects assumptions suggest a trade-off between restrictions on time and individual heterogeneity. Here we show that our sufficient conditions for Assumption PT-NSP constitute interpretable primitive conditions on the selection mechanism that imply *combinations* of time homogeneity and correlated random effects restrictions from the nonseparable panel literature.

The following assumption is the time homogeneity assumption from Chernozhukov et al. (2013) imposed on ε_{it}^{μ} in Assumption NSP-X, conditional on the time series of all covariates that enter the outcome equation.

Assumption TH. $\varepsilon_{i1}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu} \overset{d}{=} \varepsilon_{i2}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu}$

Assumption TH requires the distribution of ε_{it}^{μ} to be homogeneous across time conditional on G_i, X_i^{μ}, X_i^{λ}, and α_i^{μ}. However, it does not impose any restrictions on the conditional distribution of ε_{it}^{μ}. Furthermore, there are no restrictions imposed on the distribution of $\alpha_i^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}$, consistent with the notion of fixed effects.

The next assumption is a nonparametric correlated random effects assumption (e.g., Altonji and Matzkin, 2005; Ghanem, 2017).

Assumption CRE. $(\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|G_i, X_i^{\mu}, X_i^{\lambda} \stackrel{d}{=} (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$.

Assumption CRE is a conditional independence condition between G_i and the unobservables that enter the time-varying component of the structural function, $\lambda_t(\cdot)$. This assumption does not imply conditional random assignment, $(Y_{i1}(0), Y_{i2}(0)) \perp\!\!\!\perp G_i|X_i^{\mu}, X_i^{\lambda}$, since selection into treatment can depend on the unobservables entering the time-invariant component $\mu(\cdot)$.

Together, Assumptions TH and CRE imply Assumption PT-NSP.[18]

Proposition 6.2 (Assumptions TH and CRE imply Assumption PT-NSP)**.** *Suppose that Assumption NSP-X holds and* $P(G_i = 1|X_{i1}^{\mu} = X_{i2}^{\mu}, X_i^{\lambda}) \in (0, 1)$ *a.s. Then Assumptions TH and CRE imply Assumption PT-NSP.*

In view of Proposition 6.2, it is interesting to explore the connection between selection, time homogeneity, and correlated random effects in the nonseparable DiD framework. To this end, Proposition 6.3 shows that Assumptions SC1-NSP and SC3-NSP are primitive sufficient conditions on the selection mechanism for the nonseparable model satisfying Assumptions TH and CRE.[19]

Proposition 6.3 (Connection between selection, time homogeneity, and correlated random effects)**.** *Suppose that Assumption NSP-X holds and* $G_i = g(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})$. *Then (i) Assumption SC1-NSP with* $g(\cdot)$ *in lieu of* $\bar{g}(\cdot)$ *implies Assumptions TH and CRE if* $P(G_i = 1|X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu}) \in (0, 1)$ *a.s., (ii) Assumption SC3-NSP with* $g(\cdot)$ *in lieu of* $\bar{g}(\cdot)$ *implies Assumptions TH and CRE.*

Proposition 6.3 demonstrates how restrictions on selection can be used to justify combinations of Assumptions TH and CRE.

[18] Ghanem (2017, Appendix B) discusses the nonparametric identification of the ATT through DiD either through time homogeneity or random effects assumptions.

[19] In the context of correlated random coefficient models, Graham and Powell (2012) impose a similar structure on their model.

7 Empirical illustration of sensitivity analysis

7.1 Setup and DiD analysis

We revisit the analysis of the causal effect of the NSW labor training programs on post-treatment earnings (e.g., LaLonde, 1986). We use the same dataset as Sant'Anna and Zhao (2020a) and consider the "Dehejia and Wahba (1999, 2002) sample."[20] This sample combines the experimental treatment group (185 individuals) with an observational control group (15,992 individuals).

The outcome of interest is earnings. We observe data on earnings for two pre-treatment periods, 1974 and 1975, and one post-treatment period, 1978. We also have access to a set of baseline covariates: age, years of education, and indicators for high school dropouts, married individuals, Black and Hispanic individuals.

The unconditional DiD estimate using period 1975 as the pre-treatment period ($t = 1$) and 1978 as the post-treatment period ($t = 2$) is equal to $\widehat{\text{DiD}} = 3{,}621$ (s.e. 610). A comparison to the experimental benchmark, which is 1,794 (s.e. 671), shows that the unconditional DiD substantially overestimates the returns to the training program.

With covariates, the regression-adjusted DiD estimate under Assumption PT-X is equal to $E_n[\widehat{\text{DiD}}(X_i)|G_i = 1] = 2{,}436$ (s.e. 654), where E_n denotes the sample average and $\widehat{\text{DiD}}(X_i)$ is the conditional regression-adjusted DiD estimate.[21] This shows that adjusting for differences in baseline covariates can substantially reduce the bias of unconditional DiD relative to the experimental benchmark.

When additional pre-treatment periods are available, researchers typically report pre-tests for parallel trends in support of DiD. Based on the pre-treatment data from 1974 and 1975, the unconditional and regression-adjusted DiD estimates are 197 (s.e. 280) and 335 (s.e. 309), respectively.

Despite the non-rejections of the pre-trend tests, the sensitivity of the DiD estimates to parallel trends violations remains a major concern for two reasons. First, these tests can be substantially underpowered (e.g., Roth, 2022). Second, pre-tests are, by construction, not direct tests of Assumptions PT and PT-X. Our approach to sensitivity analysis addresses this concern and allows us to incorporate contextual knowledge about selection into the analysis.

[20] The data are from the `DRDID` R-package (Sant'Anna and Zhao, 2020b).

[21] See Appendix A for a detailed description of the regression-adjusted DiD estimator.

7.2 Sensitivity analysis

Selection on the information in the pre-treatment period is a common concern when evaluating training programs (e.g., Ashenfelter dip). Our necessary conditions demonstrate that for parallel trends to hold when selection is based on pre-treatment unobservables, the untreated potential outcome has to satisfy a martingale property. The violation of this martingale property threatens the validity of DiD in our application. We therefore assess the sensitivity of the empirical results to deviations from the martingale property using our approach.

We start by illustrating the sensitivity analysis without covariates. We assume that the deviation from the martingale assumption is linear. Replacing the population expectations by sample averages in (9) yields the following plug-in estimate of $\text{ATT}(\rho_2)$,

$$\begin{aligned}\widehat{\text{ATT}}(\rho_2) &= \widehat{\text{DiD}} - (\rho_2 - 1)(E_n[Y_{i1}|G_i = 1] - E_n[Y_{i1}|G_i = 0]), \\ &= 3{,}621 - (\rho_2 - 1)(-12{,}119).\end{aligned}$$

Average earnings in 1975 are much lower in the treatment than in the control group, leading to a substantial selection bias.

Because the impact of the pre-treatment selection bias on $\widehat{\text{ATT}}(\rho_2)$ is linear in ρ_2, even small changes in the deviation from the martingale assumption result in substantial changes in $\widehat{\text{ATT}}(\rho_2)$. Figure 1a illustrates this lack of robustness by plotting $\widehat{\text{ATT}}(\rho_2)$ as a function of ρ_2, including standard errors.[22] In our application, we have access to outcome data from two pre-treatment periods, 1974 and 1975. Therefore, it is helpful to estimate ρ_1, which parametrizes the deviation from the martingale property in the pre-treatment period, as a benchmark.[23] The estimate equals $\hat{\rho}_1 = 0.603$ and is depicted in Figure 1a.

Overall, the sensitivity analysis without covariates shows that the estimated ATT is very sensitive to deviations from the martingale property. The lack of robustness is driven by the treatment and control groups being very different before the treatment. This discussion suggests that we may reduce the selection bias in the pre-treatment period and improve the robustness of our results by adjusting for differences in baseline covariates.

Motivated by this discussion, we incorporate covariates into our sensitivity analysis. In Appendix A, we show that under a linear relaxation of the conditional martingale property,

[22] The formula for standard errors is a special case of the corresponding formula with covariates, which is given in Appendix A.

[23] Recall that the post-treatment earnings are measured in 1978, so that ρ_2 measures the persistence over three years. To account for the difference in periodicity, we proceed in two steps. First, we regress $\dot{Y}_{i1975}$ on $\dot{Y}_{i1974}$ to obtain an estimate of the yearly persistence in the pre-treatment period, $\tilde{\rho}_1 = 0.845$. Second, we adjust for the difference in periodicity by computing $\hat{\rho}_1$ as $\hat{\rho}_1 = (\hat{\tilde{\rho}}_1)^3 = 0.603$. This is justified under a linear AR(1) model for the demeaned outcomes in the pre-treatment period, $\dot{Y}_{is} = \tilde{\rho}_1 \dot{Y}_{i(s-1)} + \xi_{is},\ i = 1, \dots, n,\ s = 1972, 1973, 1974, 1975$.

the unconditional ATT is identified as

$$\text{ATT}(\rho_2) = E[\text{DiD}(X_i)|G_i = 1] - (\rho_2 - 1)(E[Y_{i1}|G_i = 1] - E[E[Y_{i1}|G_i = 0, X_i]|G_i = 1]).$$

The first term is the ATT estimand under Assumption PT-X, and the second term measures the selection bias in the pre-treatment period after adjusting for covariates. This suggests the following plug-in estimator

$$\widehat{\text{ATT}}(\rho_2) = E_n[\widehat{\text{DiD}}(X_i)|G_i = 1] - (\rho_2 - 1)(E_n[Y_{i1}|G_i = 1] - E_n[\widehat{m}_{10}(X_i)|G_i = 1]),$$

where $\widehat{m}_{10}(X_i)$ is an estimator of $E[Y_{i1}|G_i = 0, X_i]$. Using the regression-based estimators described in Appendix A, we find that

$$\widehat{\text{ATT}}(\rho_2) = 2{,}436 - (\rho_2 - 1)(-6{,}113).$$

Adjusting for differences in baseline covariates reduces the magnitude of the selection bias by approximately 50%. As a result, incorporating covariates makes the ATT less sensitive to violations of the martingale property. Figure 1b illustrates the reduced sensitivity by plotting $\widehat{\text{ATT}}(\rho_2)$ as a function of ρ_2 on the same scale as in Figure 1a. The standard errors are computed using the formula in Appendix A. The estimate of ρ_1 with covariates is $\widehat{\rho}_1 = 0.566$, which is somewhat smaller than without covariates.[24]

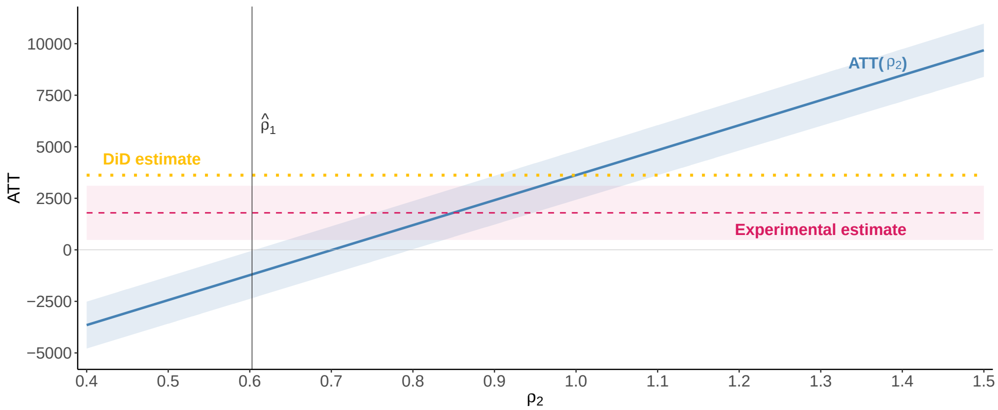
The empirical application in this section shows how the selection-based approach to sensitivity analysis can be used to assess the sensitivity of empirical results. A key practical takeaway of our analysis is that because the ATT is a linear function of the selection bias, reducing the selection bias by incorporating baseline covariates is crucial for making empirical results robust to violations of the martingale property necessary for parallel trends under imperfect foresight.

8 Conclusion and implications for practice

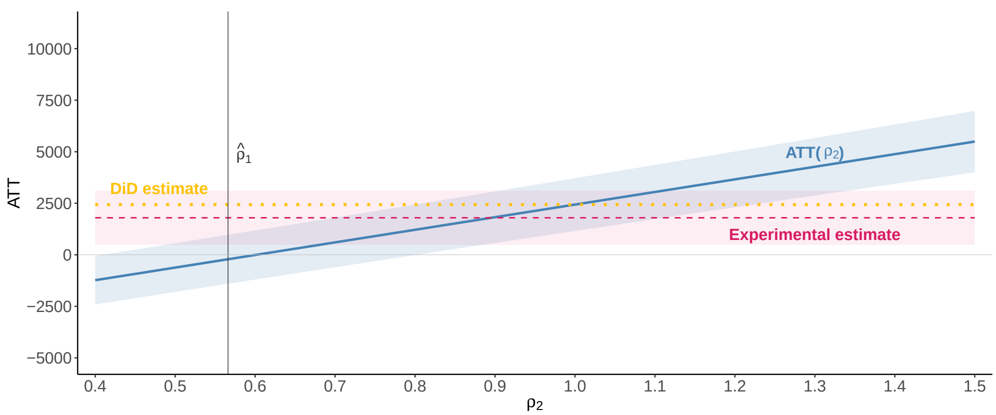
In this paper, we study popular parallel trends assumptions through the lens of selection into treatment. We derive necessary and sufficient conditions that clarify the empirical content of parallel trends, suggest selection-based approaches to sensitivity analysis, and provide theory-based templates for justifying parallel trends in applications with and without covariates. Below, we summarize the main implications of our results for practitioners.

[24]Under the linear relaxation of the martingale assumption, the yearly persistence in the pre-treatment period, $\tilde{\rho}_1$, can be estimated by regressing $\ddot{Y}_{i1975}$ on $\ddot{Y}_{i1974}$. The resulting estimate is $\hat{\tilde{\rho}}_1 = 0.827$. Adjusting for the difference in periodicity yields $\widehat{\rho}_1 = (\hat{\tilde{\rho}}_1)^3 = 0.566$.

Figure 1: Sensitivity analysis



(a) Sensitivity analysis without covariates



(b) Sensitivity analysis with covariates

Notes: Figure 1a displays the results from the sensitivity analysis without covariates. Figure 1b shows the results from the sensitivity analysis with regression adjustment. The shaded areas depict 95% confidence intervals. Detailed descriptions of the estimators and formulas for the standard errors are given in Appendix A. Data: Sant'Anna and Zhao (2020b).

Restrictions on selection are unavoidable in DiD designs. Our necessary and sufficient condition in Proposition 3.1 underscores that if researchers are not willing to impose any restrictions on selection, then parallel trends implies that the potential outcomes are constant over time up to deterministic location shifts. Therefore, in realistic settings, relying on parallel trends assumptions implicitly imposes restrictions on the time-varying unobservables and how selection depends on them.

Parallel trends can be compatible with selection on time-varying unobservables. It is well-understood that selection on time-invariant unobservables is compatible with parallel trends in the classical two-way fixed effects model under strict exogeneity (e.g., Blundell and Dias, 2009). The primitive sufficient conditions in Section 5 provide cases where parallel trends could hold despite selection depending on time-invariant *and* time-varying unobservables. An important implication is that parallel trends can be compatible with selection on untreated potential outcomes (Example 2.1) and selection on treatment effects (Example 2.2).

Assumptions on selection are useful for sensitivity analyses. Assumptions on selection into treatment are useful for performing sensitivity analysis in applications where the validity of the parallel trends assumption is questionable. To illustrate, we characterize the ATT under imperfect foresight in settings where the martingale assumptions necessary for parallel trends may be violated. For applications where contextual knowledge about selection is available, this characterization is useful for performing sensitivity analysis, computing bounds on the ATT, and developing robust inference procedures.

Contextual knowledge about selection can be used to justify parallel trends. The menu of primitive sufficient conditions in Section 5 provides practitioners with explicit theory-based templates for justifying parallel trends. These conditions consist of different combinations of restrictions on (i) which/how unobservables determine selection and (ii) how their distribution varies over time. We recommend that empirical researchers relying on these conditions use contextual information to assess and explicitly discuss which determinants of the untreated potential outcome affect selection. In doing so, it is crucial to consider the timing of the decision as well as the information set available to the units.[25] Once a suitable selection mechanism is identified, the next step is to discuss the plausibility of the corresponding assumption on the distribution of the unobservables. In this context, periodicity is crucial both to distinguish between time-invariant and time-varying factors and to justify the distributional assumptions.

How to condition on covariates depends on how they enter the outcome model. If the covariates and the unobservable determinants of selection enter the outcome model separably, researchers can condition on the entire time series of covariates and identify the overall ATT. If there are time-varying covariates that interact with the unobservable determinants of selection in the outcome model, researchers should condition on these covariates not changing over time and settle for identification of the ATT for a subpopulation.

Restrictions on nonseparable outcome models can also be used to justify parallel trends. An implication of Section 6.3 is that parallel trends is consistent with a nonseparable outcome model satisfying a combination of time homogeneity and correlated random effects assumptions. This provides researchers with an alternative avenue for justifying par-

[25] The importance of the information available to units is underscored by the results in Marx et al. (2023), who study economic models of selection including learning and optimal stopping.

allel trends based on restrictions on the untreated potential outcome and its unobservable determinants.

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Appendix (for online publication)

A Implementing sensitivity analyses with covariates

Here we describe how to incorporate covariates into the sensitivity analysis. Using the same arguments as Sections 4 and 6, the necessary condition for Assumption PT-X under imperfect foresight with one additional pre-treatment period is

$$E[\ddot{Y}_{i2}(0)|X_i, \alpha_i, \varepsilon_i^1] = \ddot{Y}_{i1}(0).$$

We consider the following relaxation of this assumption:

$$E[\ddot{Y}_{it}(0)|X_i, \alpha_i, \varepsilon_{i0}, \ldots, \varepsilon_{i(t-1)}] = \phi(\ddot{Y}_{i(t-1)}(0); \rho_t(X_i)), \quad i = 1, \ldots, n, \quad t = 1, 2.$$

The conditional ATT is identified as $\text{ATT}(X_i) = \text{DiD}(X_i) - \Delta_{\text{post}}(X_i; \rho_2(X_i))$, where

$$\Delta_{\text{post}}(X_i; \rho_2(X_i)) = \frac{E[G_i(\phi(\ddot{Y}_{i1}; \rho_2(X_i)) - \ddot{Y}_{i1})|X_i]}{P(G_i = 1|X_i)P(G_i = 0|X_i)}.$$

In Section 7, we consider the following linear relaxation of the martingale property,

$$E[\ddot{Y}_{it}(0)|X_i, \alpha_i, \varepsilon_{i0}, \ldots, \varepsilon_{i(t-1)}] = \rho_t \ddot{Y}_{i(t-1)}(0), \quad i = 1, \ldots, n, \quad t = 1, 2. \tag{12}$$

Under (12), the conditional ATT is identified as

$$\text{ATT}(X_i; \rho_2) = \text{DiD}(X_i) - (\rho_2 - 1)(E[Y_{i1}|G_i = 1, X_i] - E[Y_{i1}|G_i = 0, X_i]),$$

and, consequently, the unconditional ATT is identified as

$$\text{ATT}(\rho_2) = E[\text{DiD}(X_i)|G_i = 1] - (\rho_2 - 1)(E[Y_{i1}|G_i = 1] - E[E[Y_{i1}|G_i = 0, X_i]|G_i = 1]). \tag{13}$$

The term $E[\text{DiD}(X_i)|G_i = 1]$ is the DiD estimand under Assumption PT-X (see Section 6). The term $E[Y_{i1}|G_i = 1] - E[E[Y_{i1}|G_i = 0, X_i]|G_i = 1]$ measures the selection bias in the pre-treatment period after adjusting for covariate differences.

Because $\text{ATT}(\rho_2)$ in (13) is the difference between two standard estimands, estimation and inference can proceed based on well-established methods. Here we use a regression-based approach. Alternatively, one could use propensity-score, doubly robust, or double ML methods. Specifically, we consider the following estimator,

$$\widehat{\text{ATT}}(\rho_2) = E_n[\widehat{\text{DiD}}(X_i)|G_i = 1] - (\rho_2 - 1)(E_n[Y_{i1}|G_i = 1] - E_n[\widehat{m}_{10}(X_i)|G_i = 1]),$$

where, for a generic A_i, $E_n[A_i|G_i = 1] = \sum_{i=1}^n G_i A_i / \sum_{i=1}^n G_i$ is the sample mean of A_i among treated units, $\widehat{m}_{t0}(x) = P(x)'\widehat{\theta}_{t0}$ is an estimator of $E[Y_{it}|G_i = 0, X_i = x]$, with $P(x)$ being a known vector of transformations of x, and $E_n[\widehat{\text{DiD}}(X_i)|G_i = 1]$ is the regression-adjusted DiD estimator,

$$E_n[\widehat{\text{DiD}}(X_i)|G_i = 1] = E_n[Y_{i2} - Y_{i1}|G_i = 1] - E_n[\widehat{m}_{\Delta 0}(X_i)|G_i = 1],$$

where $\widehat{m}_{\Delta 0}(X_i) = P(x)'\widehat{\theta}_{\Delta 0}$ as an estimator of $E[Y_{i2} - Y_{i1}|G_i = 0, X_i = x]$. In our application, we estimate all regression coefficients using ordinary least squares, and $P(X_i)$ includes an intercept, linear terms for all covariates (age, years of education, and indicators for high school dropouts, married individuals, Black and Hispanic individuals), age squared, age cubed, and years of schooling squared. This specification is similar to the one in Dehejia and Wahba (1999, 2002) and Sant'Anna and Zhao (2020a), except that we omit terms related to lagged outcomes.

Conducting inference here is relatively straightforward as we can leverage results for parametric two-step estimators available in Newey and McFadden (1994) and Sant'Anna and Zhao (2020a) in a DiD context. More specifically, under mild smoothness and moment conditions, as discussed in Appendix A of Sant'Anna and Zhao (2020a), we can leverage the delta method to establish the asymptotic linear representation of $\sqrt{n}\left(\widehat{\mathrm{ATT}}(\rho_2) - \mathrm{ATT}(\rho_2)\right)$ for a given ρ_2 as

$$\sqrt{n}\left(\widehat{\mathrm{ATT}}(\rho_2) - \mathrm{ATT}(\rho_2)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\upsilon^{\mathrm{DiD}}(W_i,\theta_0) - (\rho_2-1)\upsilon^{\mathrm{bl}}(W_i,\theta_0)\right) + o_p(1), \quad (14)$$

where $W_i = (Y_{i1}, Y_{i2}, G_i, X_i')'$, $\theta_0 = (\theta_{\Delta 0}', \theta_{10}')'$, and $\upsilon^{\mathrm{DiD}}(W_i,\theta_0)$ and $\upsilon^{\mathrm{bl}}(W_i,\theta_0)$ are the asymptotic linear representation of $\sqrt{n}\left(E_n[\widehat{\mathrm{DiD}}(X_i)|G_i=1] - E[\mathrm{DiD}(X_i)|G_i=1]\right)$, and $\sqrt{n}\left((E_n[Y_{i1}|G_i=1] - E_n[\widehat{m}_{10}(X_i)|G_i=1]) - (E[Y_{i1}|G_i=1] - E[m_{10}(X_i)|G_i=1])\right)$, respectively, and are given by

$$\begin{aligned}\upsilon^{\mathrm{DiD}}(W_i,\theta_0) &= \upsilon_1^{\mathrm{DiD}}(W_i,\theta_0) - \upsilon_{\mathrm{est}}^{\mathrm{DiD}}(W_i,\theta_0),\\ \upsilon^{\mathrm{bl}}(W_i,\theta_0) &= \upsilon_1^{\mathrm{bl}}(W_i,\theta_0) - \upsilon_{\mathrm{est}}^{\mathrm{bl}}(W_i,\theta_0),\end{aligned}$$

where $\Delta Y_i = Y_{i2} - Y_{i1}$,

$$\begin{aligned}\upsilon_1^{\mathrm{DiD}}(W_i,\theta_0) &= \frac{G_i}{E[G_i]}\left((\Delta Y_i - P(X_i)'\theta_{\Delta 0}) - E\left[\Delta Y_i - P(X_i)'\theta_{\Delta 0}|G_i=1\right]\right),\\ \upsilon_{\mathrm{est}}^{\mathrm{DiD}}(W_i,\theta_0) &= E\left[P(X_i)'|G_i=1\right]E\left[P(X_i)(1-G_i)P(X_i)'\right]^{-1}(1-G_i)P(X_i)(\Delta Y_i - P(X_i)'\theta_{\Delta 0}),\\ \upsilon_1^{\mathrm{bl}}(W_i,\theta_0) &= \frac{G_i}{E[G_i]}\left((Y_1 - P(X_i)'\theta_{10}) - E\left[(Y_{i1} - P(X_i)'\theta_{10}|G_i=1\right]\right),\\ \upsilon_{\mathrm{est}}^{\mathrm{bl}}(W_i,\theta_0) &= E\left[P(X_i)'|G_i=1\right]E\left[P(X_i)(1-G_i)P(X_i)'\right]^{-1}(1-G_i)P(X_i)(Y_{i1} - P(X_i)'\theta_{10}).\end{aligned}$$

From (14) and the central limit theorem, we have that, for each ρ_2, as $n\to\infty$,

$$\sqrt{n}\left(\widehat{\mathrm{ATT}}(\rho_2) - \mathrm{ATT}(\rho_2)\right) \xrightarrow{d} N\left(0, E\left[\left(\upsilon^{\mathrm{DiD}}(W_i,\theta_0) - (\rho_2-1)\upsilon^{\mathrm{bl}}(W_i,\theta_0)\right)^2\right]\right).$$

The asymptotic variance can be estimated using its sample analog, and one can conduct inference based on it.

B Disaggregate data and aggregate decisions

In some DiD applications, the data is available at the disaggregate level (e.g., at the individual or firm level), while the decision to select into the treatment is made at the aggregate level (e.g., at the county or state level). The results in the main text directly apply to such settings by interpreting i as indexing the aggregate unit making the selection decision and the unobservables and potential outcomes as aggregate quantities. However, to justify restrictions about selection into treatment, it can be helpful to be more explicit about how selection at the aggregate level is related to the disaggregate level. In the following, we provide a formal framework for doing so. A leading example is when aggregate decisions are based on aggregating preferences at the disaggregate level (e.g., based on voting mechanisms).

Consider a 2×2 DiD setting with S groups, indexed by $s \in \{1, \dots, S\}$. Each group contains n_s units, indexed by $i \in \{1, \dots n_s\}$. To simplify the exposition, suppose that all groups are the same size, $n_s = n$ for $s \in \{1, \dots, S\}$. Following the analysis in the main text, we impose general nonseparable models for the disaggregate potential outcomes,

$$Y_{ist}(0) = \xi_{st}(\alpha_{is}, \varepsilon_{ist}).$$

The aggregate potential outcomes are given by

$$Y_{st}(0) = A_{Y(0)}(Y_{1st}(0), \dots, Y_{nst}(0)),$$

where $A_{Y(0)}(\cdot)$ is a potentially nonlinear aggregation function that can depend on n. A simple example is when the aggregate outcomes are averages of the disaggregate outcomes, $Y_{st}(0) = n^{-1} \sum_{i=1}^{n} Y_{ist}(0)$.

We consider a sharp DiD setting in which the treatment decisions are made at the group level, so that $G_s = G_{is}$ for all $i \in \{1, \dots, n\}$, and researchers rely on parallel trends at the group level,

$$E[Y_{s2}(0) - Y_{s1}(0)|G_s = 1] = E[Y_{s2}(0) - Y_{s1}(0)|G_s = 0]. \tag{15}$$

The aggregate selection decision can depend on all unit-level unobservables,

$$G_s = g(\alpha_s, \varepsilon_{s1}, \varepsilon_{s2}, \nu_s, \eta_{s1}, \eta_{s2}), \tag{16}$$

where $\alpha_s = (\alpha_{1s}, \dots, \alpha_{ns})$, $\varepsilon_{s1} = (\varepsilon_{1s1}, \dots, \varepsilon_{ns1})$, and $\varepsilon_{s2} = (\varepsilon_{1s2}, \dots, \varepsilon_{ns2})$. The vectors $\nu_s = (\nu_{1s}, \dots, \nu_{ns})$, $\eta_{s1} = (\eta_{1s1}, \dots, \eta_{ns1})$, and $\eta_{s2} = (\eta_{1s2}, \dots, \eta_{ns2})$ contain additional time-invariant and time-varying unobservables.

All results in the main text directly apply in this setting with i replaced by s, such that there are no additional theoretical complications. However, being explicit about the disaggregate level can help "microfound" restrictions on the aggregate selection mechanism $g(\cdot)$, as we illustrate in the following example.

Example B.1 (Simple majority voting)**.** *Suppose that the aggregate selection decision is* *based on simple majority voting. Each unit submits a vote* $V_{is} \in \{0,1\}$*, where*

$$V_{is} = v(\alpha_{is}, \varepsilon_{is1}, \varepsilon_{is2}, \nu_{is}, \eta_{is1}, \eta_{is2}). \tag{17}$$

The voting mechanism (17) accommodates voting based on group-level unobservables and outcomes since the additional unobservables $(\nu_{is}, \eta_{is1}, \eta_{is2})$ *are unrestricted and can contain group-level quantities. Votes can be based on potential outcomes, expected gains, and fixed effects (as in Examples 2.1, 2.2, and 2.3), or other considerations.*

The aggregate selection decision under simple majority voting is

$$G_s = 1\left\{\frac{1}{n}\sum_{i=1}^{n} V_{is} \geq 0.5\right\}. \tag{18}$$

This selection mechanism is a special case of mechanism (16). Restrictions on the aggregate mechanism (18) can be directly motivated based on assumptions on the units' voting behavior, their information sets, and discount factors. □

C Necessary and sufficient conditions: extensions

C.1 Parallel trends for any distribution

In the main text, we derive necessary and sufficient conditions for a scenario where researchers are not willing to choose a specific selection mechanism. Here we consider an alternative scenario where researchers are not willing to impose any restrictions on the distribution of unobservables, $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$, and require parallel trends to hold for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$.

The following proposition shows that Assumption PT holds for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$ in a complete class if and only if selection is independent of the time-invariant and time-varying unobservable determinants of $Y_{it}(0)$. Before we state the proposition, we recall the definition of a complete class of distributions (Equations (4.8)–(4.9) on p.115 in Lehmann and Romano, 2005).

Definition C.1 (Completeness of a class of distributions)**.** *Let* W *be a vector of random variables. A family of distributions* $\mathcal{F}$ *is complete if*

$$E[f(W)] = 0 \quad \textit{for all} \quad F_W \in \mathcal{F}$$

implies

$$f(w) = 0 \quad \textit{almost everywhere (a.e.)} \quad \mathcal{F}.$$

Proposition C.1 (Necessary and sufficient condition for parallel trends for any distribution of unobservables)**.** *Suppose that* $g \in \mathcal{G}_{all}$ *and* $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$, *where* $\mathcal{F}$ *is a complete family of probability distributions satisfying* $P(g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1) \in (0, 1)$ *and* $P(\dot{Y}_{i1}(0) \neq \dot{Y}_{i2}(0)) = 1$. *Assumption PT holds for all* $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$ *if and only if* $P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1)$ *a.s. for all* $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$.

In Proposition C.1, we require $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$ to belong to a complete family of distributions, $\mathcal{F}$. Completeness requires that the class of possible distributions of unobservables is rich enough. This condition is key for showing that parallel trends implies that selection is independent of all unobservable determinants of $Y_{i1}(0)$ and $Y_{i2}(0)$. It holds automatically when $\mathcal{F}$ is unrestricted.

C.2 Multiple periods and multiple groups

Here we generalize our results to DiD designs with multiple periods and multiple groups. The setup and notation are based on Callaway and Sant'Anna (2021), Sun and Abraham (2021), and Roth et al. (2023).

Let $t \in \{1, 2, \ldots, T\}$ index the periods. Suppose that at time $t = 1$, no units are treated, at $t = 2$, some units become treated, while others remain untreated, and so on. Previously treated units remain treated for all periods. Units can be categorized based on their treatment adoption pattern $D_i = (D_{i1}, \ldots, D_{iT})$. We define the group indicator G_i as the first period in which units are treated, $G_i = \min\{t \in \{1, \ldots, T\} : D_{it} = 1\}$, and set $G_i = \infty$ for the never-treated units so that $G_i \in \{2, \ldots, T, \infty\}$.[26]

Potential outcomes are indexed by the entire treatment sequence $(d_1, \ldots, d_T) \in \{0, 1\}^T$, $Y_{it}(d_1, \ldots, d_T)$. Since treatment is an absorbing state, the potential outcomes can be indexed by the first treatment period only. Define $Y_{it}(g) = Y_{it}(\mathbf{0}_{g-1}, \mathbf{1}_{T-g+1})$ for $g \in \{2, \ldots, T\}$ and $Y_{it}(\infty) = Y_{it}(\mathbf{0}_T)$, where $\mathbf{0}_s \equiv (0, \ldots, 0) \in \mathbb{R}^s$ and $\mathbf{1}_s \equiv (1, \ldots, 1) \in \mathbb{R}^s$. Observed outcomes are given by $Y_{it} = \sum_{g\in\{2,\ldots,T,\infty\}} 1\{G_i = g\}Y_{it}(g)$. We maintain a standard no-anticipation assumption (e.g., Roth et al., 2023).

Assumption NA. *For* $g \in \{2, \ldots, T, \infty\}$ *and* $t < g$, $Y_{it}(g) = Y_{it}(\infty)$.

Our objects of interest are the group-time ATTs,

$$\text{ATT}(g, t) = E[Y_{it}(g) - Y_{it}(\infty)|G_i = g]. \tag{19}$$

We impose the following parallel trends assumption to identify the $\text{ATT}(g, t)$.[27]

[26] Since G_i is a random variable with finite support, we emphasize that $\{\infty\}$ is merely a label.

[27] In our setting, this parallel trends assumption corresponds to the ones made by Callaway and Sant'Anna

Assumption PT-MP. *For* $(g,t) \in \{2,\ldots,T\}^2$

$$E[Y_{it}(\infty) - Y_{i(t-1)}(\infty) \,|\, G_i = g] = E[Y_{it}(\infty) - Y_{i(t-1)}(\infty) | G_i = \infty] \tag{20}$$

We consider a general nonseparable outcome model,

$$Y_{it}(\infty) = \xi_t(\alpha_i, \varepsilon_{it}), \quad i = 1,\ldots,n, \quad t = 1,\ldots,T.$$

Selection into treatment can depend on the unobservable determinants of $Y_{it}(\infty)$ as well as additional unobservables,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \nu_i, \eta_{i1}, \ldots, \eta_{iT}).$$

As before, let $\mathcal{G}_{\text{all}}$ denote the set of all selection mechanisms $g(\cdot)$ and define the following classes of restricted selection mechanisms, which are natural analogs of those considered in Section 3.

$$\mathcal{G}_{\text{if}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \ldots, e_T, v, t_1, \ldots, t_T) \text{ is a trivial function of } (e_2, \ldots, e_T, t_2, \ldots, t_T)\}$$
$$\mathcal{G}_{\text{fe}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \ldots, e_T, v, t_1, \ldots, t_T) \text{ is a trivial function of } (e_1, \ldots, e_T, t_1, \ldots, t_T)\}$$

The following assumption generalizes Assumption SEL to the multiple-period, multiple-group setting. It ensures that the selection mechanisms used to establish the necessary and sufficient conditions for parallel trends are non-degenerate.

Assumption SEL-MP. *There exists a component of* ν_i*, labeled* ν_i^1 *(w.l.o.g.), such that* $\nu_i^1 \perp\!\!\!\perp (\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT})$*. In addition, there exists a non-overlapping partition of the support of* v_i^1*,* $\{B_g\}_{g=2}^T$*, such that* $P(v_i^1 \in B_g) \in (0,1)$ *for* $g \in \{2,\ldots,T\}$*.*

The following three propositions extend the necessary and sufficient conditions in Propositions 3.1, 3.2, and 3.3 to the more general DiD setting in this section. All these conditions are natural generalizations of their counterparts in the 2×2 case.

Proposition C.2 (Necessary and sufficient condition for $g \in \mathcal{G}_{\text{all}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ *or* $P(\dot{Y}_{it}(\infty) < \dot{Y}_{i(t-1)}(\infty)) < 1$ *for each* $t \in \{2,\ldots,T\}$*. Then Assumption PT-MP holds for all* $g \in \mathcal{G}_{all}$ *satisfying* $P(G_i = g) \in (0,1)$ *for* $g \in \{2,\ldots,T,\infty\}$ *if and only if* $\dot{Y}_{i1}(\infty) = \cdots = \dot{Y}_{iT}(\infty)$ *a.s.*

(2021), Gardner (2021), Sun and Abraham (2021), Wooldridge (2021), and Borusyak et al. (2023); see also de Chaisemartin and D'Haultfœuille (2020) and Marcus and Sant'Anna (2021) for related assumptions.

Proposition C.3 (Necessary condition for $g \in \mathcal{G}_{\text{if}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ *or* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] < E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ *for each* $t \in \{2, \ldots, T\}$*. If Assumption PT-MP holds for all* $g \in \mathcal{G}_{if}$ *satisfying* $P(G_i = g) \in (0, 1)$ *for* $g \in \{2, \ldots, T, \infty\}$*, then* $E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]$ *a.s. for* $t \in \{2, \ldots, T\}$*.*

Proposition C.4 (Necessary condition for $g \in \mathcal{G}_{\text{fe}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(E[\dot{Y}_{it}(\infty)|\alpha_i] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ *or* $P(E[\dot{Y}_{it}(\infty)|\alpha_i] < E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ *for each* $t \in \{2, \ldots, T\}$*. If Assumption PT-MP holds for all* $g \in \mathcal{G}_{fe}$ *satisfying* $P(G_i = g) \in (0, 1)$ *for* $g \in \{2, \ldots, T, \infty\}$*, then* $E[\dot{Y}_{it}(\infty)|\alpha_i] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]$ *a.s. for* $t \in \{2, \ldots, T\}$*.*

The necessary conditions in Propositions C.3 and C.4 are sufficient for PT-MP under straightforward extensions of the conditions in Section 3.2.

D Sensitivity analysis: multiple periods and multiple groups

Here we outline how the sensitivity analysis we propose in Section 4 can be extended to the multiple-period, multiple-group case.

We first restate the necessary condition for $\mathcal{G}_{\text{if}}$ in Proposition C.3 as follows

$$E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(\infty) \text{ a.s. for } t = 2, \ldots, T.$$

In the presence of additional pre-treatment periods, $t = -T_{\text{pre}}, -(T_{\text{pre}} + 1), \ldots, 0$, the necessary condition generalizes to

$$E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i(-T_{\text{pre}})}, \ldots, \varepsilon_{i0}, \varepsilon_{i1}] = \dot{Y}_{i1}(\infty) \text{ for } t = 2, \ldots, T.$$

This restatement of the martingale property suggests the following multiple-period counterpart of Assumption REL.

Assumption REL-MP. *The following relaxation of the martingale condition holds for* $\tau \in \mathbb{N}^+$*:*

$$E[\dot{Y}_{i(t+\tau)}(\infty)|\alpha_i, \varepsilon_{i(-T_{pre})}, \ldots, \varepsilon_{it}] = \phi(\dot{Y}_{it}(\infty); \rho_{t,\tau}),\ i = 1, \ldots, n,\ t = -T_{pre}, \ldots, T - \tau,$$

where $\phi(\cdot; \rho_{t,\tau})$ *is a function that is known up to the parameter* $\rho_{t,\tau}$*, which may be infinite dimensional.*

Under imperfect foresight, Assumption REL-MP, and additional regularity conditions, we can characterize the ATT(g, t) as a function of $\rho_{t,\tau}$ using similar arguments as in Proposition 4.1.

E Templates for justifying Assumption PT-X

Consider the following separable model with covariates.

Assumption SP-X.

$$Y_{it}(0) = \alpha_i + \lambda_t + \gamma_t(X_{it}) + \varepsilon_{it}, \quad E[\varepsilon_{it}] = 0, \quad i = 1, \ldots, n, \quad t = 1, 2. \tag{21}$$

Assumption SP-X allows for nonparametric covariate-specific trends, which is a key reason for incorporating covariates in DiD analyses. It nests commonly used parametric specifications such as $\gamma_t(X_{it}) = X_{it}'\beta_t$. We assume that the treatment does not affect X_{it}.

To focus on the different roles played by the time-varying observable and unobservable determinants of $Y_{it}(0)$, we state our sufficient conditions in terms of the projected selection mechanism,

$$\bar{g}(x_1, x_2, a, e_1, e_2) = E[G_i | X_{i1} = x_1, X_{i2} = x_2, \alpha_i = a, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2].$$

Assumption SC1-X. *The following conditions hold:*

(i) $\bar{g}(x_1, x_2, a, e_1, e_2)$ *is a symmetric function in* e_1 *and* e_2.

(ii) $\varepsilon_{i1}, \varepsilon_{i2} | X_i, \alpha_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1} | X_i, \alpha_i$.

Assumption SC2-X. *The following conditions hold:*

(i) $\bar{g}(x_1, x_2, a, e_1, e_2)$ *is a trivial function of* e_2.

(ii) $E[\varepsilon_{i2} - \varepsilon_{i1} | X_i, \alpha_i, \varepsilon_{i1}] = E[\varepsilon_{i2} - \varepsilon_{i1} | X_i]$.

Assumption SC3-X. *The following conditions hold:*

(i) $\bar{g}(x_1, x_2, a, e_1, e_2)$ *is a trivial function of* e_1 *and* e_2.

(ii) $E[\varepsilon_{i1} | X_i, \alpha_i] = E[\varepsilon_{i2} | X_i, \alpha_i]$.

Assumptions SC1-X, SC2-X, and SC3-X are conditional versions of Assumptions SC1, SC2, and SC3. They demonstrate that incorporating time-varying covariates makes the restrictions on the selection mechanism more plausible.

The following proposition shows that Assumptions SC1-X, SC2-X, and SC3-X are sufficient for Assumption PT-X.

Proposition E.1 (Templates for justifying Assumption PT-X)**.** *Suppose that Assumption SP-X holds and* $P(G_i = 1|X_i) \in (0, 1)$ *a.s. Then (i) Assumption SC1-X implies Assumption PT-X, (ii) Assumption SC2-X implies Assumption PT-X, and (iii) Assumption SC3-X implies Assumption PT-X.*

Proposition E.1 provides theory-based templates for justifying Assumption PT-X in applications based on contextual knowledge about selection into treatment.

F Proofs of the results in the main text

F.1 Auxiliary lemmas

Lemma F.1. *Let* ω_i *denote a vector of random variables. Suppose that* $P(G_i = 1|\omega_i) \in (0, 1)$ *a.s. Then* $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]$ *if and only if* $E[G_i(Y_{i2}(0) - Y_{i1}(0))|\omega_i] = E[G_i|\omega_i]E[Y_{i2}(0) - Y_{i1}(0)|\omega_i]$ *a.s.*

Proof. In the following, all equalities involving conditional expectations are understood as a.s. equalities.

"$\Longrightarrow$": First, note that by the law of total probability, $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]$ implies

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|\omega_i].$$

The result follows from noting that $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = \frac{E[G_i(Y_{i2}(0) - Y_{i1}(0))|\omega_i]}{P(G_i=1|\omega_i)}$ by definition.

"$\Longleftarrow$": Since $P(G_i = 1|\omega_i) \in (0, 1)$, it follows that $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|\omega_i]$. It then follows that

$$\begin{aligned}
&E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i]P(G_i = 1|\omega_i) + E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]P(G_i = 0|\omega_i) \\
&= E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i].
\end{aligned}$$

The result follows from subtracting the first term on the left-hand side and dividing by $P(G_i = 0|\omega_i)$. □

Lemma F.2. *For a scalar random variable* W_i*, let* $\dot{W}_i = W_i - E[W_i]$*. If* $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$ *or* $E[\dot{W}_i 1\{\dot{W}_i \geq 0\}] = 0$*, then* $W_i = E[W_i]$ *a.s.*

Proof. We prove the results for the case where $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$, since the proof for the other case follows by identical arguments. First, note that by definition $E[\dot{W}_i] = 0$, which is equivalent to

$$E[\dot{W}_i^+] = E[\dot{W}_i^-], \tag{22}$$

where $\dot{W}_i^+ = |\dot{W}_i| 1\{\dot{W}_i > 0\}$ and $\dot{W}_i^- = |\dot{W}_i| 1\{\dot{W}_i < 0\}$.

Now suppose that $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$ holds, which is equivalent to

$$E[\dot{W}_i^+ 1\{\dot{W}_i \leq 0\}] = E[\dot{W}_i^- 1\{\dot{W}_i \leq 0\}], \tag{23}$$

since, by definition, $\dot{W}_i = \dot{W}_i^+ - \dot{W}_i^-$. Note that the left-hand side equals zero by the definition of $\dot{W}_i^+$. As a result, $E[\dot{W}_i^- 1\{\dot{W}_i \leq 0\}] = E[\dot{W}_i^-] = 0$. Since $\dot{W}_i^- \geq 0$, this implies that $P(\dot{W}_i^- = 0) = 1$. Now note that $P(\dot{W}_i^- = 0) = P(|\dot{W}_i| 1\{\dot{W}_i < 0\} = 0) = P(1\{\dot{W}_i < 0\} = 0) = 1$, which implies $P(\dot{W}_i < 0) = 0$.

Since $E[\dot{W}_i] = 0$, (22) further implies that $E[\dot{W}_i^-] = E[\dot{W}_i^+] = 0$. Since $\dot{W}_i^+ \geq 0$, it follows that $P(\dot{W}_i^+ = 0) = 1$. Now note that $P(\dot{W}_i^+ = 0) = P(|\dot{W}_i| 1\{\dot{W}_i > 0\} = 0) = P(1\{\dot{W}_i > 0\} = 0) = 1$, which implies $P(\dot{W}_i > 0) = 0$.

Together, $P(\dot{W}_i < 0) = 0$ and $P(\dot{W}_i > 0) = 0$ imply that $P(\dot{W}_i = 0) = 1 - (P(\dot{W}_i < 0) + P(\dot{W}_i > 0)) = 1$, which completes the proof. □

Lemma F.3. *Let ω_i denote a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$. Suppose that $P(\upsilon_i^1 > c) \in (0,1)$ for some $c \in \mathbb{R}$, and $\nu_i^1 \perp\!\!\!\perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$.*

(i) If $P(E[\dot{Y}_{i2}(0)|\omega_i] > E[\dot{Y}_{i1}(0)|\omega_i]) < 1$ and Assumption PT holds for $G_i = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}$, then $E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] = 0$ a.s.

(ii) If $P(E[\dot{Y}_{i2}(0)|\omega_i] < E[\dot{Y}_{i1}(0)|\omega_i]) < 1$ and Assumption PT holds for $G_i = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \geq 0\}$, then $E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] = 0$ a.s.

Proof. We only prove (i). The proof of (ii) follows from the same arguments. Under the maintained assumptions the selection mechanism is nondegenerate,

$$P(1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\} = 1) \in (0,1).$$

Thus, by Lemma F.1, Assumption PT holding for $G_i = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}$ is equivalent to

$$E[1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0,$$

which, by $P(v_i^1 > c) \in (0,1)$ and $\nu_i^1 \perp\!\!\!\perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$, is equivalent to

$$E[1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0.$$

By the law of iterated expectations (LIE), this is further equivalent to

$$E[1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i]] = 0$$

Since $E[E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i]] = 0$, the result follows by Lemma F.2. □

Lemma F.4. *Let* $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ *denote a vector of random variables. Suppose that* $\varepsilon_{i1}, \varepsilon_{i2}|\alpha_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1}|\alpha_i$ *holds. Then*

(i) $F_{\varepsilon_{i1}|\alpha_i}(e|a) = F_{\varepsilon_{i2}|\alpha_i}(e|a)$ *a.e.* $(a, e) \in \mathcal{A} \times \mathcal{E}$

(ii) $F_{\varepsilon_{i1}|\varepsilon_{i2},\alpha_i}(e_1|e_2, a) = F_{\varepsilon_{i2}|\varepsilon_{i1},\alpha_i}(e_1|e_2, a)$ *a.e.* $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$.

Proof. **(i)** By the definition of the marginal distribution, the conditional exchangeability restriction implies (i) by the following *a.e.*

$$F_{\varepsilon_{i1}|\alpha_i}(e_1|a) = \lim_{e_2 \to \infty} F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1, e_2|a) = \lim_{e_2 \to \infty} F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_2, e_1|a) = F_{\varepsilon_{i2}|\alpha_i}(e_1|a). \quad (24)$$

(ii) By the definition of the conditional distribution and (i) of this lemma, the conditional exchangeability restriction implies (ii) by the following

$$F_{\varepsilon_{i1}|\varepsilon_{i2},\alpha_i}(e_1|e_2, a) = \frac{F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1, e_2|a)}{F_{\varepsilon_{i2}|\alpha_i}(e_2|a)} = \frac{F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_2, e_1|a)}{F_{\varepsilon_{i1}|\alpha_i}(e_2|a)} = F_{\varepsilon_{i2}|\varepsilon_{i1},\alpha_i}(e_1|e_2, a), \quad (25)$$

a.e. $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$. □

F.2 Propositions

F.2.1 Proof of Proposition 3.1

"$\Longrightarrow$": We first consider the case where $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{all}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0) \leq 0\}$. By Assumption SEL and $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$, we can invoke Lemma F.3.i with $\omega_i = (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$, which implies $\dot{Y}_{i1}(0) = \dot{Y}_{i2}(0)$ a.s.

The proof for the case where $P(\dot{Y}_{i2}(0) < \dot{Y}_{i1}(0)) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0) \geq 0\}$ and invoking Lemma F.3.ii.

"$\Longleftarrow$": This direction is immediate. □

F.2.2 Proof of Proposition 3.2

We first consider the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{if}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}] \leq 0\}$. By Assumption SEL and $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$, we can invoke Lemma F.3.i with $\omega_i = (\alpha_i, \varepsilon_{i1})$, which implies $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ a.s.

The proof for the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] < \dot{Y}_{i1}(0)) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}] \geq 0\}$ and invoking Lemma F.3.ii. □

F.2.3 Proof of Proposition 3.3

We first consider the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{fe}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i] \leq 0\}$. By Assumption SEL and $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[Y_{i1}(0)|\alpha_i]) < 1$, we can invoke Lemma F.3.i with $\omega_i = \alpha_i$, which implies $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ a.s.

The proof for the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i] < E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i] \geq 0\}$ and invoking Lemma F.3.ii. □

F.2.4 Proof of Proposition 3.4

(i) Since $g \in \mathcal{G}_{\text{if}}$, then we can simplify the following expression by the LIE as follows,

$$
\begin{aligned}
E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\
&= E[E[E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))|\alpha_i, \varepsilon_{i1}]] \\
&= E[E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}]].
\end{aligned}
$$

The third equality follows from $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$.[28] If $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ a.s., then the last term equals zero, which implies the result by Lemma F.1.

(ii) Similar to (i), since $g \in \mathcal{G}_{\text{fe}}$ and $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} \nu_i|\alpha_i$, then we can simplify the following

[28]The conditional independence restriction specifically implies the following:

$$
\begin{aligned}
&E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i = a, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2] = \int g(a, e_1, v, t_1) dF_{\nu_i, \eta_{i1}|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1|a, e_1, e_2) \\
&= \int g(\alpha_i, \varepsilon_{i1}, v, t_1) dF_{\nu_i, \eta_{i1}|\alpha_i, \varepsilon_{i1}}(v, t_1|a, e_1) = E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i = a, \varepsilon_{i1} = e_1].
\end{aligned}
$$

expression by the LIE as follows,

$$\begin{aligned} E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[g(\alpha_i, \nu_i)(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\ &= E[E[E[g(\alpha_i, \nu_i)|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))|\alpha_i]] \\ &= E[E[g(\alpha_i, \nu_i)|\alpha_i]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i]]. \end{aligned}$$

If $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ a.s., then the last term equals zero, which implies the result by Lemma F.1. □

F.2.5 Proof of Proposition 4.1

First, we simplify Δ_{post} as follows

$$\begin{aligned} \Delta_{\text{post}} &= \frac{E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))]}{P(G_i = 1)} - \frac{E[(1 - G_i)(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))]}{P(G_i = 0)} \\ &= \frac{E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))]}{P(G_i = 1)P(G_i = 0)} \end{aligned} \tag{26}$$

By the LIE, it then follows that

$$\begin{aligned} \Delta_{\text{post}} &= \frac{E[E[E[G_i|\alpha_i, \varepsilon_i^1, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))|\alpha_i, \varepsilon_i^1]]}{P(G_i = 0)P(G_i = 1)} \\ &= \frac{E[E[G_i|\alpha_i, \varepsilon_i^1]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_i^1]]}{P(G_i = 0)P(G_i = 1)} \\ &= \frac{E[E[G_i|\alpha_i, \varepsilon_i^1](\phi(\dot{Y}_{i1}(0); \rho_2) - \dot{Y}_{i1}(0))]}{P(G_i = 0)P(G_i = 1)} \\ &= \frac{E[G_i(\phi(\dot{Y}_{i1}; \rho_2) - \dot{Y}_{i1})]}{P(G_i = 0)P(G_i = 1)}. \end{aligned} \tag{27}$$

The second equality follows from $E[G_i|\alpha_i, \varepsilon_i^1, \varepsilon_{i2}] = E[G_i|\alpha_i, \varepsilon_i^1]$, which is an implication of Assumption IF. The third equality follows from Assumption REL. The last equality follows by the LIE and because $Y_{i1} = Y_{i1}(0)$. The result follows from the definition of the ATT. □

F.2.6 Proof of Proposition 5.1

(i) We first show that (i) and (iii) of Assumption SC1 imply the symmetry of $\bar{g}(a, e_1, e_2) = E[G_i|\alpha_i = a, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2]$ in e_1 and e_2. To do so, we note that these two conditions imply the following for $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$

$$\bar{g}(a, e_1, e_2) = \int g(a, e_1, e_2, v, t_1, t_2) dF_{\nu_i, \eta_{i1}, \eta_{i2} | \alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1, t_2 | a, e_1, e_2)$$
$$= \int g(a, e_2, e_1, v, t_1, t_2) dF_{\nu_i, \eta_{i1}, \eta_{i2} | \alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1, t_2 | a, e_2, e_1) = \bar{g}(a, e_2, e_1), \tag{28}$$

where the penultimate equality follows by the symmetry of $g(\cdot)$ and $F_{\nu_i, \eta_{i1}, \eta_{i2} | \alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}$ in e_1 and e_2 imposed in (i) and (iii) in Assumption SC1, respectively.

Next, by the LIE, we can decompose $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})]$ and then invoke the symmetry restrictions on $\bar{g}(\cdot)$ and $F_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}$ implied by (i) and (iii) of Assumption SC1 as well as (ii) of Assumption SC1, respectively:

$$E[G_i(\varepsilon_{i2} - \varepsilon_{i1})] = E[E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) \varepsilon_{i2} | \alpha_i] - E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) \varepsilon_{i1} | \alpha_i]]$$
$$= \int \left(\int \bar{g}(a, e_1, e_2) e_2 dF_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}(e_1, e_2 | a) - \int \bar{g}(a, e_1, e_2) e_1 dF_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}(e_1, e_2 | a) \right) dF_{\alpha_i}(a)$$
$$= \int \left(\int \bar{g}(a, e_2, e_1) e_2 dF_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}(e_2, e_1 | a) - \int \bar{g}(a, e_1, e_2) e_1 dF_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}(e_1, e_2 | a) \right) dF_{\alpha_i}(a) = 0.$$

The second equality follows from the symmetry restrictions on $\bar{g}(\cdot)$ and $F_{\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i}$. Together, they imply that both conditional expectations in the parentheses equal $E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) \varepsilon_{i1} | \alpha_i]$, and therefore the difference between them is zero. As a result, Assumption SC1 implies Assumption PT.

(ii) This result follows from the proof of Proposition 3.4.i by plugging-in $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$ for $t = 1, 2$.

(iii) This result follows from the proof of Proposition 3.4.ii by plugging-in $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$ for $t = 1, 2$. □

F.2.7 Proof of Proposition 6.1

In this proof, all equalities involving random variables are understood to hold a.s.

First, by Lemma F.1, Assumption PT-NSP under Assumption NSP-X holds if and only if

$$E[G_i(Y_{i2}(0) - Y_{i1}(0)) | X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]$$
$$= E[G_i | X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] E[Y_{i2}(0) - Y_{i1}(0) | X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]. \tag{29}$$

Next, we state some preliminary observations and then proceed to show each statement separately.

Note that, by the LIE, Assumption SEL-CI and the definition of $\bar{g}(\cdot)$, the LHS of (29) equals the following

$$\begin{aligned} & E[G_i(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ =& E[E[G_i|X_i^\mu, X_i^\lambda, \alpha_i^\mu, \alpha_i^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda](Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ =& E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]. \end{aligned} \tag{30}$$

Similarly, by the LIE, the RHS of (29) equals the following,

$$\begin{aligned} & E[G_i|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ =& E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \end{aligned} \tag{31}$$

As a result, in the following, to show that Assumptions SC1-NSP, SC2-NSP, and SC3-NSP are sufficient for Assumption PT-NSP, it suffices to show that each assumption implies the following equality,

$$\begin{aligned} & E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ =& E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \end{aligned}$$

(i) By Assumption NSP-X, it follows that

$$\begin{aligned} & E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ =& E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\ & + E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu], \end{aligned} \tag{32}$$

We first examine the first term on the RHS of the above equality. Note that by the symmetry restrictions in Assumptions SC1-NSP.i and SC1-NSP.ii, it follows that a.e. $(a, x^\mu, x_1^\lambda, x_2^\lambda) \in \mathcal{A} \times \mathcal{X}_\mu \times \mathcal{X}_\lambda^2$

$$\begin{aligned} & E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)\mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu)|X_i^\lambda = (x_1^\lambda, x_2^\lambda), X_{i1}^\mu = X_{i2}^\mu = x^\mu, \alpha_i^\mu = a] \\ =& \int \bar{g}(x^\mu, x^\mu, x_1^\lambda, x_2^\lambda, a, e_1, e_2)\mu(x^\mu, a, e_1) dF_{\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu}(e_1, e_2|(x_1^\lambda, x_2^\lambda), x^\mu, a) \\ =& \int \bar{g}(x^\mu, x^\mu, x_1^\lambda, x_2^\lambda, a, e_2, e_1)\mu(x^\mu, a, e_1) dF_{\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu}(e_2, e_1|(x_1^\lambda, x_2^\lambda), x^\mu, a) \\ =& E[\bar{g}(X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \alpha_i^\mu, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda = (x_1^\lambda, x_2^\lambda), X_{i1}^\mu = X_{i2}^\mu = x^\mu, \alpha_i^\mu = a]. \end{aligned} \tag{33}$$

As a result, the first summand in (32) equals zero by (33) and the LIE.

Next, we consider the second summand in (32),

$$
\begin{aligned}
&E[\bar{g}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})(\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\bar{g}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\bar{g}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[Y_{i2}(0) - Y_{i1}(0)|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]. \qquad (34)
\end{aligned}
$$

The first equality follows from the conditional independence assumption in Assumption SC1-NSP.iii. The last equality follows from the time homogeneity of $F_{\varepsilon_{it}^{\mu}|X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu}}$, which follows from the exchangeability restriction in Assumption SC1-NSP.ii by Lemma F.4, and implies that $E[\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}, \alpha_i^{\mu}] = 0$ and

$$
E[Y_{i2}(0) - Y_{i1}(0)|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] = E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]
$$

by the LIE. As a result, the above implies that Assumption PT-NSP holds.

(ii) By Assumption SC2-NSP.i, we can define $\breve{\bar{g}}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}, e_1^{\lambda}) = \bar{g}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}, e_1^{\lambda}, e_2^{\lambda})$. By Assumption NSP-X, it follows that

$$
\begin{aligned}
&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})(Y_{i2}(0) - Y_{i1}(0))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})\Delta_{\mu,i}|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
&+ E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})(\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[\Delta_{\mu,i}|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
&+ E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[Y_{i2}(0) - Y_{i1}(0)|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \qquad (35)
\end{aligned}
$$

The second equality follows from the conditional independence conditions in Assumptions SC2-NSP.ii and SC2-NSP.iii. The last equality follows from Assumption NSP-X. Equation (35) then implies Assumption PT-NSP.

(iii) By Assumption SC3-NSP.i, we can define $\breve{\bar{g}}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}) = \bar{g}(x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, a^{\mu}, e_1^{\lambda}, e_2^{\lambda})$. Now by the Assumption NSP-X and SC3-NSP.i, it follows that

$$
\begin{aligned}
&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})(Y_{i2}(0) - Y_{i1}(0))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})(\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu}))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
&+ E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})(\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}))|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})E[\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}, \alpha_i^{\mu}]|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
&+ E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\breve{\bar{g}}(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]E[Y_{i2}(0) - Y_{i1}(0)|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}],
\end{aligned}
$$

where the first equality follows from Assumption NSP-X. The second equality follows by applying the LIE to the first term and the conditional independence imposed in Assumption SC3-NSP.iii to the second term. The first term on the RHS of the second equality equals zero by the conditioning on $X_{i1}^{\mu} = X_{i2}^{\mu}$ and the time homogeneity condition in Assumption SC3-NSP.ii. The last equality follows from noting, similar as in the proof of (i), that since $E[\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}, \alpha_i^{\mu}] = 0$,

$$E[Y_{i2}(0) - Y_{i1}(0)|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] = E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]$$

by the LIE. This completes the proof. □

F.2.8 Proof of Proposition 6.2

Under Assumption NSP-X,

$$\begin{aligned}
&E[Y_{i2}(0) - Y_{i1}(0)|G_i, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \\
=&E[\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|G_i, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \qquad (36)\\
&+ E[\lambda_2(X_{i2}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i2}^{\lambda}) - \lambda_1(X_{i1}^{\lambda}, \alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda})|G_i, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]. \qquad (37)
\end{aligned}$$

The remainder of the proof follows in two steps. First, we show that the term in (36) equals zero under our assumptions. Second, we show that the second term is conditionally mean independent of G_i, which implies Assumption PT-NSP.

We proceed to show that under Assumption TH the term in (36) equals zero by the following,

$$\begin{aligned}
&E[\mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})|G_i = g, X_i^{\lambda} = (x_1^{\lambda}, x_2^{\lambda}), X_{i1}^{\mu} = X_{i2}^{\mu} = x^{\mu}] \\
=&\int \mu(x^{\mu}, a^{\mu}, e^{\mu}) dF_{\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}}(a^{\mu}, e^{\mu}|g, (x^{\mu}, x^{\mu}), (x_1^{\lambda}, x_2^{\lambda})) \\
=&\int \mu(x^{\mu}, a^{\mu}, e^{\mu}) dF_{\alpha_i^{\mu}, \varepsilon_{i2}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}}(a^{\mu}, e^{\mu}|g, (x^{\mu}, x^{\mu}), (x_1^{\lambda}, x_2^{\lambda})) \\
=&E[\mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu})|G_i = g, X_i^{\lambda} = (x_1^{\lambda}, x_2^{\lambda}), X_{i1}^{\mu} = X_{i2}^{\mu} = x^{\mu}], \qquad (38)
\end{aligned}$$

where the first and last equalities follow by definition, whereas the penultimate equality follows from Assumption TH noting that it implies $\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda} \overset{d}{=} \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}$.

Finally, we show that Assumption CRE implies the following for (37)

$$\begin{aligned}
&E[\lambda_2(X_{i2}^\lambda,\alpha_i^\lambda,\varepsilon_{i2}^\lambda)-\lambda_1(X_{i1}^\lambda,\alpha_i^\lambda,\varepsilon_{i1}^\lambda)|G_i=g,X_i^\lambda=(x_1^\lambda,x_2^\lambda),X_{i1}^\mu=X_{i2}^\mu=x^\mu]\\
=&\int(\lambda_2(x_2^\lambda,a^\lambda,e_2^\lambda)-\lambda_1(x_1^\lambda,a^\lambda,e_1^\lambda))dF_{\alpha_i^\lambda,\varepsilon_{i1}^\lambda,\varepsilon_{i2}^\lambda|G_i,X_i^\mu,X_i^\lambda}(a^\lambda,e_1^\lambda,e_2^\lambda|g,(x^\mu,x^\mu),(x_1^\lambda,x_2^\lambda))\\
=&\int(\lambda_2(x_2^\lambda,a^\lambda,e_2^\lambda)-\lambda_1(x_1^\lambda,a^\lambda,e_1^\lambda))dF_{\alpha_i^\lambda,\varepsilon_{i1}^\lambda,\varepsilon_{i2}^\lambda|X_i^\mu,X_i^\lambda}(a^\lambda,e_1^\lambda,e_2^\lambda|(x^\mu,x^\mu),(x_1^\lambda,x_2^\lambda))\\
=&E[\lambda_2(X_{i2}^\lambda,\alpha_i^\lambda,\varepsilon_{i2}^\lambda)-\lambda_1(X_{i1}^\lambda,\alpha_i^\lambda,\varepsilon_{i1}^\lambda)|X_i^\lambda=(x_1^\lambda,x_2^\lambda),X_{i1}^\mu=X_{i2}^\mu=x^\mu],
\end{aligned}\tag{39}$$

where the penultimate equality follows by Assumption CRE. This completes the proof. □

F.2.9 Proof of Proposition 6.3

Throughout this proof, equalities involving conditioning statements are understood to hold *a.e.* We proceed to show each result separately.

(i) It suffices to show (i.a) Assumptions SC1-NSP.i and SC1-NSP.ii imply Assumption TH and (i.b) Assumptions SC1-NSP.i and SC1-NSP.iii imply Assumption CRE.

(i.a) Consider

$$F_{\varepsilon_{i1}^\mu,G_i|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1,g|x^\mu,x^\lambda,a)=F_{G_i|\varepsilon_{i1}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(g|e_1,x^\mu,x^\lambda,a)F_{\varepsilon_{i1}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1|x^\mu,x^\mu,a),$$

where $x^\mu=(x_1^\mu,x_2^\mu)$ and $x^\lambda=(x_1^\lambda,x_2^\lambda)$. Assumption SC1-NSP.ii implies $F_{\varepsilon_{i1}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e|x^\mu,x^\lambda,a)=F_{\varepsilon_{i2}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e|x^\mu,x^\lambda,a)$ as well as $F_{\varepsilon_{i1}^\mu|\varepsilon_{i2}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1|e_2,x^\mu,x^\lambda,a)=F_{\varepsilon_{i2}^\mu|\varepsilon_{i1}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1|e_2,x^\mu,x^\lambda,a)$ by Lemma F.4, which implies

$$\begin{aligned}
&F_{G_i|\varepsilon_{i1}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(g|e_1,x^\mu,x^\lambda,a)\\
=&\int 1\{g(x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda,a,e_1,e_2)\le g\}dF_{\varepsilon_{i2}^\mu|\varepsilon_{i1}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_2|e_1,x^\mu,x^\lambda,a)\\
=&\int 1\{g(x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda,a,e_2,e_1)\le g\}dF_{\varepsilon_{i1}^\mu|\varepsilon_{i2}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_2|e_1,x^\mu,x^\lambda,a)\\
=&F_{G_i|\varepsilon_{i2}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(g|e_1,x^\mu,x^\lambda,a).
\end{aligned}\tag{40}$$

As a result,

$$\begin{aligned}
&F_{\varepsilon_{i1}^\mu,G_i|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1,g|x^\mu,x^\lambda,a)=F_{G_i|\varepsilon_{i1}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(g|e_1,x^\mu,x^\lambda,a)F_{\varepsilon_{i1}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1|x^\mu,x^\lambda,a)\\
=&F_{G_i|\varepsilon_{i2}^\mu,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(g|e_1,x^\mu,x^\lambda,a)F_{\varepsilon_{i2}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1|x^\mu,x^\lambda,a)\\
=&F_{\varepsilon_{i2}^\mu,G_i|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e_1,g|x^\mu,x^\lambda,a).
\end{aligned}\tag{41}$$

This implies Assumption TH by the definition of a conditional distribution,

$$F_{\varepsilon_{it}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|g,x^{\mu},x^{\lambda},a) = \frac{F_{\varepsilon_{it}^{\mu},G_i|X_i^{\mu},X_i^{\lambda}\alpha_i^{\mu}}(e,g|x^{\mu},x^{\lambda})}{F_{G_i|X_i^{\mu},X_i^{\lambda}\alpha_i^{\mu}}(g|x^{\mu},x^{\lambda},a)},$$

where $F_{G_i|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|x^{\mu},x^{\lambda},a) > 0$ a.s. for $g = 0,1$ by assumption.

(i.b) This statement follows in a straightforward manner from the definition of G_i in Assumption SC1-NSP.i and the conditional independence condition in Assumption SC1-NSP.iii which together imply Assumption CRE. This completes the proof of (i).

(ii) To show the result, it suffices to show that (ii.a) Assumptions SC3-NSP.i and SC3-NSP.ii imply Assumption TH and (ii.b) Assumptions SC3-NSP.i and SC3-NSP.iii imply Assumption CRE.

(ii.a) Under Assumptions SC3-NSP.i and SC3-NSP.ii, $G_i = g(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})$ is a degenerate random variable equaling either zero or one with probability one conditional on X_i^{μ}, X_i^{λ} and α_i^{μ}. As a result,

$$\begin{aligned}
&F_{\varepsilon_{it}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|g,x^{\mu},x^{\lambda},a)\\
&= \sum_{g=0,1} P(\varepsilon_{it}^{\mu} \leq e|G_i = g(x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},a), X_i^{\mu} = x^{\mu}, X_i^{\lambda} = x^{\lambda}, \alpha_i^{\mu} = a)1\{g(x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},a) = g\}\\
&= \sum_{g=0,1} P(\varepsilon_{it}^{\mu} \leq e|X_i^{\mu} = x^{\mu}, X_i^{\lambda} = x^{\lambda}, \alpha_i^{\mu} = a)1\{g(x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},a) = g\}\\
&= \sum_{g=0,1} F_{\varepsilon_{it}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|a,x^{\mu},x^{\lambda})1\{g(x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},a) = g\}. \qquad (42)
\end{aligned}$$

As a result, Assumption SC3-NSP.i together with the time homogeneity of $F_{\varepsilon_{it}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}$ in Assumption SC3-NSP.ii is sufficient for the time homogeneity of $F_{\varepsilon_{it}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}$, which yields Assumption TH.

(ii.b) The statement (ii.b) is immediate from noting that Assumption SC3-NSP.iii together with $G_i = g(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu})$ imply that $g(X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \alpha_i^{\mu}) \perp\!\!\!\perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$, which is equivalent to Assumption CRE. This completes the proof of (ii). □

G Proofs of results in the Appendix

G.1 Auxiliary lemma

Lemma G.1 (Equivalence with multiple periods)**.** *Suppose that Assumption NA holds and* $P(G_i = g) \in (0,1)$ *for* $g \in \{2,\ldots,T,\infty\}$*. Then Assumption PT-MP is equivalent to* $E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))] = 0$ *for* $g \in \{2,\ldots,T,\infty\}$ *and* $t \in \{2,\ldots,T\}$*.*

Proof. Assumption PT-MP is equivalent to

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = \infty] \quad \text{for } (g,t) \in \{2,\dots,T\}^2,$$

which, since $E[\dot{Y}_{it}(\infty)] = 0$, is also equivalent to

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = 0 \quad \text{for } (g,t) \in \{2,\dots,T,\infty\} \times \{2,\dots,T\}. \tag{43}$$

Thus, we need to show that (43) is equivalent to $E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))] = 0$ for $g \in \{2,\dots,T,\infty\}$ and $t \in \{2,\dots,T\}$. This follows because

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = \frac{E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))]}{P(G_i = g)}$$

for $(g,t) \in \{2,\dots,T,\infty\} \times \{2,\dots,T\}$, since $P(G_i = g) \in (0,1)$ for $g \in \{2,\dots,T,\infty\}$ by assumption. □

G.2 Propositions

G.2.1 Proof of Proposition C.1

"$\Longrightarrow$": By Lemma F.1, Assumption PT is equivalent to $E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0$, which in turn is equivalent to the following

$$E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))] = 0, \tag{44}$$

where $\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = E[g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) - E[G_i]|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}]$ and $\dot{\xi}_t(\alpha_i, \varepsilon_{it}) = \xi_t(\alpha_i, \varepsilon_{it}) - E[Y_{it}(0)]$ for $t = 1, 2$. The equivalence between $E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0$ and (44) follows by the LIE and subtracting $E[G_i]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)]$, noting that it equals zero by construction.

It follows that Assumption PT holding for all $F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2}} \in \mathcal{F}$ is equivalent to

$$E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))] = 0, \tag{45}$$

for all $F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2}} \in \mathcal{F}$. By completeness of $\mathcal{F}$, the last equality implies the following (Lehmann and Romano, 2005, p.115)

$$P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) = 0) = 1 \quad \text{for all } F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2}} \in \mathcal{F}. \tag{46}$$

Now note that the left-hand side of (46) can be simplified as follows,

$$\begin{aligned}
&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}) = 0) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})), \dot{\xi}_2(\alpha_i, \varepsilon_{i2}) = \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
&+ P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})), \dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))|\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = 0) = 1,
\end{aligned} \tag{47}$$

where the penultimate equality follows since $P(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) = P(\dot{Y}_{i2}(0) \neq \dot{Y}_{i1}(0)) = 1$ by assumption. As a result, by the definition of $\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$,

$$P(E[g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]) = 1 \quad \text{for all } F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}} \in \mathcal{F}. \tag{48}$$

"$\Longleftarrow$": The if statement follows by the LIE. All following statements are understood to hold for all $F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}} \in \mathcal{F}$. Note that $P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1)$ a.s. is equivalent to $E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]$ a.s. Next, the LIE implies the following equality

$$\begin{aligned}
E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\
&= E[E[G_i](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0.
\end{aligned} \tag{49}$$

The second equality follows from $E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]$ a.s. The last equality follows from $E[\dot{Y}_{it}(0)] = 0$ for $t = 1, 2$ by definition. □

G.2.2 Proof of Proposition C.2

"$\Longrightarrow$": We first consider the case where $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \{2, \ldots, T\}$. Since Assumption PT-MP holds for all $g \in \mathcal{G}_{\text{all}}$, it holds for the following selection mechanism, where $\mathcal{G}_S = \{2, \ldots, T\}$ denotes the set of switcher groups,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_S \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\zeta_i = (\nu_i, \eta_{i1}, \ldots, \eta_{iT})$. By Lemma G.1, Assumption PT-MP implies that for any $g \in \mathcal{G}_S$,

$$\begin{aligned}
&E[1\{\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = g\}(\dot{Y}_{ig}(\infty) - \dot{Y}_{i(g-1)}(\infty)] \\
&= E[1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\}(\dot{Y}_{ig}(\infty) - \dot{Y}_{i(g-1)}(\infty))] = 0.
\end{aligned}$$

By Assumption SEL-MP and the additional regularity conditions in the proposition, we can invoke Lemma F.3.i while setting $\omega_i = (\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT})$ and replacing G_i $(t \in \{1,2\})$ with $1\{G_i = g\}$ $(t \in \{g-1, g\})$ for each $g \in \mathcal{G}_S$. This implies that $\dot{Y}_{ig}(0) = \dot{Y}_{i(g-1)}(0)$ a.s. for each $g \in \mathcal{G}_S = \{2, \ldots, T\}$, which implies the result.

The proof for the case where $P(\dot{Y}_{it}(\infty) < \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \{2, \ldots, T\}$ follows symmetrically using the selection mechanism,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \geq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_S \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

and invoking Lemma F.3.ii.

The proof for the case where $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \mathcal{G}_1 \subset \mathcal{G}_S$ and $P(\dot{Y}_{is}(\infty) < \dot{Y}_{i(s-1)}(\infty)) < 1$ for $s \in \mathcal{G}_2 = \mathcal{G}_1^c \cap \mathcal{G}_S$ follows from using the following selection mechanism

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_1, \\ g \text{ if } 1\{\dot{Y}_{ig}(\infty) \geq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_2, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

and invoking Lemma F.3.i for $g \in \mathcal{G}_1$ and Lemma F.3.ii for $g \in \mathcal{G}_2$.
"$\Longleftarrow$": This direction is immediate. □

G.2.3 Proof of Proposition C.3

The proof follows from similar arguments as in Proposition C.2 using the following selection mechanism for the case where $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ for $t \in \{2, \ldots, T\}$,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{E[\dot{Y}_{ig}(\infty)|\alpha_i, \varepsilon_{i1}] \leq E[\dot{Y}_{i(g-1)}(\infty)|\alpha_i, \varepsilon_{i1}]\}1\{\nu_i \in B_g\} = 1, g \in \mathcal{G}_S, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\mathcal{G}_S$ and ζ_i are defined in the proof of Proposition C.2. □

G.2.4 Proof of Proposition C.4

The proof follows from similar arguments as in Proposition C.2 using the following selection mechanism for the case where $P(E[\dot{Y}_{it}(\infty)|\alpha_i] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ for $t \in \{2, \ldots, T\}$,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{E[\dot{Y}_{ig}(\infty)|\alpha_i] \leq E[\dot{Y}_{i(g-1)}(\infty)|\alpha_i]\}1\{\nu_i \in B_g\} = 1, g \in \mathcal{G}_S, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\mathcal{G}_S$ and ζ_i are defined in the proof of Proposition C.2.

G.2.5 Proof of Proposition E.1

In this proof, all equalities involving random variables are understood to hold a.s. By Lemma F.1, it suffices to show that each assumption implies $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] = E[G_i|X_i]E[\varepsilon_{i2} - \varepsilon_{i1}|X_i]$.

(i) The exchangeability restrictions in Assumption SC1-X imply the following:

$$\begin{aligned} &E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i1}|X_i = (x_1, x_2), \alpha_i = a] \\ =& \int \bar{g}(x_1, x_2, a, e_1, e_2)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|X_i,\alpha_i}(e_1, e_2|(x_1, x_2), a \\ =& \int \bar{g}(x_1, x_2, a, e_2, e_1)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|X_i,\alpha_i}(e_2, e_1|(x_1, x_2), a) \\ =& \ E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i2}|X_i = (x_1, x_2), \alpha_i = a], \end{aligned} \tag{50}$$

a.e. $(a, x_1, x_2) \in \mathcal{A} \times \mathcal{X}^2$, where $\mathcal{X}$ denotes the support of X_{it}.

Integrating out $\alpha_i|X_i$ in the above yields the following a.e. equality:

$$\begin{aligned} &\int E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i1}|X_i = (x_1, x_2), \alpha_i = a]dF_{\alpha_i|X_i}(a|(x_1, x_2)) \\ =& \int E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i2}|X_i = (x_1, x_2), \alpha_i = a]dF_{\alpha_i|X_i}(a|(x_1, x_2)). \end{aligned} \tag{51}$$

As a result, by the LIE, we have that $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] = 0$. This completes the proof, since by Assumption SC1-X.ii $\varepsilon_{i1}|X_i \overset{d}{=} \varepsilon_{i2}|X_i$ by Lemma F.4 and therefore $E[\varepsilon_{i2} - \varepsilon_{i1}|X_i] = 0$.

(ii) Since under Assumption SC2-X, $\bar{g}(\cdot)$ is a trivial function of ε_{i2}, we can define $\check{\bar{g}}(x_1, x_2, a, e_1) = \bar{g}(x_1, x_2, a, e_1, e_2)$. Note that

$$\begin{aligned} E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] =& E[E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\varepsilon_{i2} - \varepsilon_{i1})|X_i, \alpha_i, \varepsilon_{i1}]|X_i] \\ =& E[\check{\bar{g}}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1})E[\varepsilon_{i2} - \varepsilon_{i1}|X_i, \alpha_i, \varepsilon_{i1}]|X_i] \\ =& E[\check{\bar{g}}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1})E[\varepsilon_{i2} - \varepsilon_{i1}|X_i]|X_i] \\ =& E[G_i|X_i]E[\varepsilon_{i2} - \varepsilon_{i1}|X_i], \end{aligned} \tag{52}$$

where the first equality follows by the LIE. The second equality follows from Assumption SC2-X.i. The third equality follows by Assumption SC2-X.ii, which implies the result in the last equality.

(iii) Since $\bar{g}(\cdot)$ is a trivial function of ε_{i1} and ε_{i2} under Assumption SC3-X, we can define $\breve{\bar{g}}(x_1, x_2, a) = \bar{g}(x_1, x_2, a, e_1, e_2)$.

$$\begin{aligned} E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] &= E[E[\bar{g}(X_{i1}, X_{i2}, \alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\varepsilon_{i2} - \varepsilon_{i1})|X_i, \alpha_i]|X_i] \\ &= E[\breve{\bar{g}}(X_{i1}, X_{i2}, \alpha_i)E[\varepsilon_{i2} - \varepsilon_{i1}|X_i, \alpha_i]|X_i] = 0. \end{aligned} \quad (53)$$

The first equality follows by the LIE. The second equality follows by Assumption SC3-X.i. The last equality follows from $E[\varepsilon_{i1}|X_i, \alpha_i] = E[\varepsilon_{i2}|X_i, \alpha_i]$ under Assumption SC3-X.ii. The result then follows from noting that $E[\varepsilon_{i2} - \varepsilon_{i1}|X_i] = 0$ under this assumption, which completes the proof. □